

SPECTRAL ASYMPTOTICS FOR THE DIRICHLET LAPLACIAN WITH A NEUMANN WINDOW VIA A BIRMAN-SCHWINGER ANALYSIS OF THE DIRICHLET-TO-NEUMANN OPERATOR

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ABSTRACT. In the present article we will give a new proof of the ground state asymptotics of the Dirichlet Laplacian with a Neumann window acting on functions which are defined on a two-dimensional infinite strip or a three-dimensional infinite layer. The proof is based on the analysis of the corresponding Dirichlet-to-Neumann operator as a first order classical pseudo-differential operator. Using the explicit representation of its symbol we prove an asymptotic expansion as the window length decreases.

1. INTRODUCTION

In what follows we consider an infinite quantum waveguide subject to a perturbation of the boundary conditions. In spectral theory this type of perturbation is of particular interest, since it is non-additive and may not be treated with standard methods, such as the Birman-Schwinger principle. The simplest case arises by considering the Dirichlet Laplacian on an infinite strip having a so-called Neumann window. Let $\Omega = \mathbb{R} \times (0, \alpha)$. We consider $-\Delta$ on Ω with Dirichlet boundary on all of $\partial\Omega$ except for some small part of the boundary, where we impose Neumann boundary conditions. We are interested in the behaviour of the discrete eigenvalues below the essential spectrum $[\pi^2/\alpha^2, \infty)$ depending on the length of the window, cf. Figure 1.

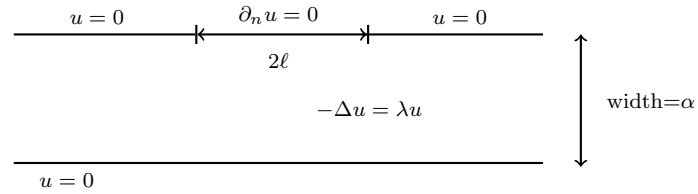


FIGURE 1. The Dirichlet Laplacian with a Neumann window.

This case was first investigated in [11], where the existence of an eigenvalue was proved by a variational argument. Moreover, a numerical computation given by these authors suggested that for small windows of size 2ℓ the distance of the eigenvalue to the spectral threshold π^2/α^2 is of order ℓ^4 . The first analytic proof concerning this fact was given by Exner and Vugalter in [12]. They proved a two-sided asymptotic estimate, i.e., for small $\ell > 0$ there exist a unique eigenvalue $\lambda(\ell)$ below the essential spectrum $[\pi^2/\alpha^2, \infty)$ and

constants $c_1, c_2 > 0$ such that

$$c_1 \ell^4 \leq \pi^2/\alpha^2 - \lambda(\ell) \leq c_2 \ell^4 \quad \text{as } \ell \rightarrow 0. \quad (1.1)$$

In fact in [11, 12] the authors considered the more general case of two quantum waveguides which are coupled through a small window, cf. Figure 2.

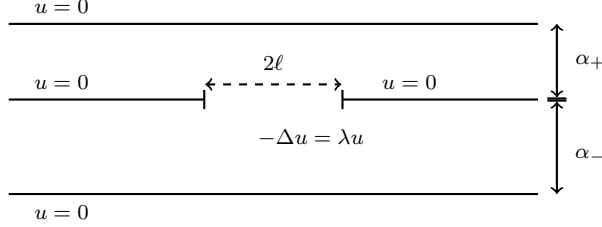


FIGURE 2. Two laterally coupled quantum waveguides of width α_+ and α_- .

If both waveguides have the same width $\alpha_+ = \alpha_- = \alpha$, then we may use the symmetry with respect to the horizontal direction. In this case the eigenvalue problem is equivalent to the mixed problem in Figure 1. The proof of the two-sided asymptotic estimate in [12] is based on a variational argument. The upper bound may easily be obtained using a suitable test-function and the min-max principle for self-adjoint operators. However, the more delicate part consists in finding a uniform lower bound for the variational coefficient. In order to prove such an estimate Exner and Vugalter decomposed an arbitrary test-function, using an expansion in the vertical direction.

Popov [27] refined the two-sided estimate and proved that the ground state eigenvalue $\lambda(\ell)$ satisfies the following asymptotic behaviour

$$\frac{\pi^2}{\alpha^2} - \lambda(\ell) = \begin{cases} \left(\frac{\pi^3}{4\alpha^3}\right)^2 \ell^4 + o(\ell^4), & \alpha_+ \neq \alpha_-, \\ \left(\frac{\pi^3}{2\alpha^3}\right)^2 \ell^4 + o(\ell^4), & \alpha_+ = \alpha_-, \end{cases} \quad \text{as } \ell \rightarrow 0. \quad (1.2)$$

His proof is based on a scheme which matches the asymptotic expansions for the eigenfunctions, cf. [23, 17]. Popov uses different expansions for the eigenfunctions near the window and distant from the window. Using the explicit formulae for the Green's functions in the upper and lower waveguide he computes the asymptotic behaviour of the eigenvalue. Further terms in this expansion have been calculated in [15]. In [30] the approach was generalised to three-dimensional layers, cf. also [13] for a two-sided asymptotic estimate. Further extensions include e.g. higher dimensional cylinders [28, 18], a finite or an infinite number of windows [28, 29, 5, 6, 7, 26], the case of three coupled waveguides [16] or magnetic operators [8]. The case of two retracting distant windows has been investigated in [9, 10]. For an overview concerning spectral problems in quantum waveguides we refer to [14].

We provide a new approach for the symmetric case which uses the explicit representation of the Dirichlet-to-Neumann operator. This allows us to reformulate the singular

perturbation of the original operator into an additive perturbation of the Dirichlet-to-Neumann operator, or merely its truncated part. We replace the matching scheme for the eigenfunctions in [27] by an asymptotic expansion of the Dirichlet-to-Neumann operator and a subsequent use of the Birman-Schwinger principle. As a particular consequence we will observe that only the principal symbol has an influence on the first term of the asymptotic formula. In a similar way we treat the case of two coupled quantum waveguides. An application of the method to elastic waveguides may be found in [22].

Structure of the article. We start by treating the two-dimensional case. We consider the Laplacian $-\Delta$ on $\Omega = \mathbb{R} \times (0, \alpha)$ with Dirichlet boundary conditions except for some small set $\Sigma_\ell \times \{0\} \subseteq \partial\Omega$, where we impose Robin boundary conditions. Here $\Sigma_\ell := \ell \cdot \Sigma \subseteq \mathbb{R}$ and $\Sigma \subseteq \mathbb{R}$ is a finite union of bounded open intervals. Section 2 starts with the definition of the self-adjoint realisation of the corresponding Laplacian and the introduction of the Dirichlet-to-Neumann operator and of the Dirichlet-to-Robin operator. The asymptotic formula for the eigenvalue of the corresponding Laplacian is stated and proven in Section 2, see Theorems 2.7 and 2.8. Additionally, in Theorems 2.7 and 2.8 we prove the uniqueness of the eigenvalue for small window sizes and in Theorem 2.12 we treat the case of two quantum waveguides coupled through a small window.

Section 3 is devoted to three-dimensional layers of the form $\Omega = \mathbb{R}^2 \times (0, \alpha)$. In this case the Robin window is given by $\Sigma_\ell \times \{0\} \subseteq \partial\Omega$, where $\Sigma_\ell := \ell \cdot \Sigma \subseteq \mathbb{R}^2$ is a bounded open set with Lipschitz boundary, cf. Figure 3. We follow the same scheme as in the two-dimensional case and prove in Theorem 3.1 an asymptotic formula for the ground state eigenvalue as the window length decreases.

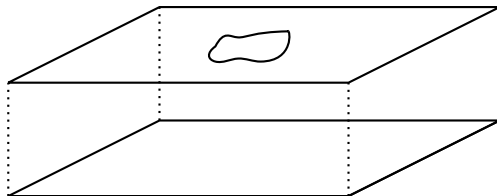


FIGURE 3. An infinite layer with a small window.

2. THE TWO-DIMENSIONAL CASE

The construction of the Dirichlet-to-Robin operator. Let $\alpha > 0$ and put $\Omega := \mathbb{R} \times (0, \alpha)$ with coordinates $(x, y) \in \Omega$. Let $\Sigma \subseteq \mathbb{R}$ be the Robin window and assume that Σ is a bounded open set, which is a finite union of open intervals. We denote the scaled window by $\Sigma_\ell := \ell \cdot \Sigma$. The Laplacian on Ω with Robin boundary conditions on $\Sigma_\ell \times \{0\}$ and Dirichlet boundary conditions on the remaining part of the boundary is defined by the quadratic form

$$a_{\ell,b}[u] := \int_{\Omega} |\nabla u(x, y)|^2 \, dx \, dy + \int_{\Sigma_\ell} b(x) \cdot |u(x, 0)|^2 \, dx, \quad (2.1)$$

with the form domain

$$D[a_{\ell,b}] := \{u \in H^1(\Omega) : u|_{\mathbb{R} \times \{\alpha\}} = 0 \wedge \text{supp}(u|_{\mathbb{R} \times \{0\}}) \subseteq \overline{\Sigma_\ell}\}. \quad (2.2)$$

Here $b \in L_\infty(\mathbb{R})$ is a real-valued function and $u|_{\mathbb{R} \times \{0\}}, u|_{\mathbb{R} \times \{\alpha\}} \in H^{1/2}(\mathbb{R})$ are the boundary traces of the function $u \in H^1(\Omega)$. Then $a_{\ell,b}$ defines a closed semi-bounded form on $L_2(\Omega)$ and gives rise to a self-adjoint operator, which we denote by $A_{\ell,b}$. The essential spectrum of the operator $A_{\ell,b}$ is given by $\sigma_{\text{ess}}(A_{\ell,b}) = [\pi^2/\alpha^2, \infty)$. This well-known fact is due to Birman [3], where he gives a proof in the case of compact boundary $\partial\Omega$.

As a first step we consider the Dirichlet-to-Neumann operator acting on the lower part of the boundary $\mathbb{R} \times \{0\}$. For $s \in \mathbb{R}$ let $H^s(\mathbb{R})$ be the standard Sobolev space on \mathbb{R} with the usual norm defined via Fourier transform. Let $\omega \in \mathbb{C}$ and $g \in H^{1/2}(\mathbb{R})$. We consider a weak solution $u \in H^1(\Omega)$ of the Poisson problem

$$(-\Delta - \omega)u = 0 \quad \text{in } \Omega, \quad u|_{\mathbb{R} \times \{0\}} = g, \quad u|_{\mathbb{R} \times \{\alpha\}} = 0. \quad (2.3)$$

Applying the Fourier transform in the horizontal direction, it follows from (2.3) that $\hat{u}(\xi, y)$ solves

$$(-\partial_y^2 + \xi^2 - \omega)\hat{u}(\xi, y) = 0, \quad \hat{u}(\xi, 0) = \hat{g}(\xi), \quad \hat{u}(\xi, \alpha) = 0 \quad (2.4)$$

with $(\xi, y) \in \mathbb{R} \times (0, \alpha)$. Conversely, if $u \in H^1(\Omega)$ is given such that its Fourier transform $\hat{u}(\xi, y)$ solves the family (2.4) of Sturm Liouville problems, then u is a weak solution of the Poisson problem (2.3). For fixed $\xi \in \mathbb{R}$ with $\xi^2 \neq \omega$, the solution of (2.4) is given by

$$\hat{u}(\xi, y) = \frac{\hat{g}(\xi)}{\sinh(\alpha\sqrt{\xi^2 - \omega})} \cdot \sinh((\alpha - y)\sqrt{\xi^2 - \omega}). \quad (2.5)$$

Here and subsequently we choose the branch of the square root function such that $z \mapsto \sqrt{z}$ is holomorphic in $\mathbb{C} \setminus (-\infty, 0]$ and such that $\sqrt{z} > 0$ for $z > 0$. Moreover, we extend the definition to $z \in (-\infty, 0]$ and assume that $\text{Im}(\sqrt{z}) \geq 0$ for $z \leq 0$. Actually, the expression for \hat{u} is independent of the value of the square root function as long as one uses the same in the two terms.

Lemma 2.1. *Let $\omega \in \mathbb{C} \setminus [\pi^2/\alpha^2, \infty)$. For every $g \in H^{1/2}(\mathbb{R})$ there exists a unique $u \in H^1(\Omega)$ which solves (2.3), and moreover $\|u\|_{H^1(\Omega)} \leq c\|g\|_{H^{1/2}(\mathbb{R})}$ with $c = c(\omega, \alpha) > 0$ independent of g .*

For the proof of Lemma 2.1 one has to verify that the function u given by (2.5) belongs indeed to $H^1(\Omega)$ if $g \in H^{1/2}(\mathbb{R})$. We want to omit this simple calculation.

Remark. If $\omega \geq \pi^2/\alpha^2$, then in general \hat{u} will have a singularity and the above mapping property does not hold true. This is to be expected, as in this case ω will be located in the essential spectrum $[\pi^2/\alpha^2, \infty)$.

Here and subsequently we assume that $\omega \in \mathbb{C} \setminus [\pi^2/\alpha^2, \infty)$. Let $g \in H^{1/2}(\mathbb{R})$ and let u be the solution of the Poisson problem (2.3). Its normal derivative $\partial_n u$ satisfies

$$\widehat{\partial_n u}(\xi, 0) = m_\omega(\xi)\hat{g}(\xi),$$

where

$$m_\omega(\xi) := \sqrt{\xi^2 - \omega} \cdot \coth(\alpha\sqrt{\xi^2 - \omega}). \quad (2.6)$$

The Dirichlet-to-Neumann operator is defined by $D_\omega : H^{1/2}(\mathbb{R}) \rightarrow H^{-1/2}(\mathbb{R})$,

$$\widehat{D_\omega g}(\xi) := \hat{g}(\xi) \cdot \sqrt{\xi^2 - \omega} \cdot \coth(\alpha \sqrt{\xi^2 - \omega}). \quad (2.7)$$

We note that D_ω is a classical pseudo-differential operator of order 1 with x -independent symbol m_ω . Since the operator $A_{\ell,b}$ is defined by its quadratic form we give a variational characterisation of the Dirichlet-to-Neumann operator D_ω . We also refer to Chapter 4 in McLean [25] for mixed problems formulated in their variational form.

Lemma 2.2. *Let $\omega \in \mathbb{C} \setminus [\pi^2/\alpha^2, \infty)$ and $g \in H^{1/2}(\mathbb{R})$. We denote by $u \in H^1(\Omega)$ the solution of the Poisson problem (2.3). Then for $h \in H^{-1/2}(\mathbb{R})$ the following two assertions are equivalent:*

- (1) $h = D_\omega g$.
- (2) For all $v \in H^1(\Omega)$ with $v|_{\mathbb{R} \times \{\alpha\}} = 0$ we have

$$\langle \nabla u, \nabla v \rangle_\Omega = \omega \langle u, v \rangle_\Omega + \langle h, v|_{\mathbb{R} \times \{0\}} \rangle_{\mathbb{R}}. \quad (2.8)$$

Here $\langle \cdot, \cdot \rangle_\Omega$ and $\langle \cdot, \cdot \rangle_{\mathbb{R}}$ denote the dual pairings with respect to the scalar product in $L_2(\Omega)$ and $L_2(\mathbb{R})$ identified with $L_2(\mathbb{R} \times \{0\})$.

Proof. Let $g \in H^{1/2}(\mathbb{R})$ and $u \in H^1(\Omega)$ be chosen as above. From (2.5) and integration by parts we obtain

$$\begin{aligned} \langle \nabla u, \nabla v \rangle_\Omega - \omega \langle u, v \rangle_\Omega &= \int_{\mathbb{R}} \partial_y \hat{u}(\xi, 0) \overline{\hat{v}(\xi, 0)} \, d\xi \\ &= \int_{\mathbb{R}} \sqrt{\xi^2 - \omega} \cdot \coth(\alpha \sqrt{\xi^2 - \omega}) \hat{g}(\xi) \overline{\hat{v}(\xi, 0)} \, d\xi \\ &= \langle D_\omega g, v|_{\mathbb{R} \times \{0\}} \rangle_{\mathbb{R}}. \end{aligned}$$

This proves one direction of the equivalence. The converse follows as the trace operator $u \mapsto u|_{\mathbb{R} \times \{0\}} : H^1(\Omega) \rightarrow H^{1/2}(\mathbb{R})$ has a continuous right inverse, cf. [25, Lemma 3.36]. In particular, for every $f \in H^{1/2}(\mathbb{R})$ there exists $v \in H^1(\Omega)$ such that $v|_{\mathbb{R} \times \{\alpha\}} = 0$ and $v|_{\mathbb{R} \times \{0\}} = f$, and thus, $D_\omega g$ is uniquely defined by (2.8). \square

In order to treat the mixed boundary value problem we introduce for $s \in \mathbb{R}$ the following function spaces

$$\tilde{H}_0^s(\Sigma_\ell) := \{g \in H^s(\mathbb{R}) : \text{supp}(g) \subseteq \overline{\Sigma_\ell}\}, \quad (2.9)$$

$$H^s(\Sigma_\ell) := \{g \in (C_c^\infty(\Sigma_\ell))' : \exists G \in H^s(\mathbb{R}) \text{ with } g = G|_{\Sigma_\ell}\}. \quad (2.10)$$

Here $C_c^\infty(\Sigma_\ell)$ is the space of smooth functions with compact support in Σ_ℓ ; we denote by $(C_c^\infty(\Sigma_\ell))'$ the space of distributions on Σ_ℓ . We note that $\tilde{H}_0^s(\Sigma_\ell)$ is a closed subspace of distributions in \mathbb{R} whereas $H^s(\Sigma_\ell)$ is a subspace of distributions in Σ_ℓ . The latter space may be identified with the quotient space

$$H^s(\mathbb{R}) / \tilde{H}_0^s(\mathbb{R} \setminus \overline{\Sigma_\ell}),$$

where $\tilde{H}_0^s(\mathbb{R} \setminus \overline{\Sigma_\ell})$ contains, by definition, those distributions in $H^s(\mathbb{R})$ which have support in $\mathbb{R} \setminus \Sigma_\ell$. We endow the spaces in (2.9) and (2.10) with their natural topology, i.e., $\tilde{H}_0^s(\Sigma_\ell)$ carries the subspace topology of $H^s(\mathbb{R})$ and $H^s(\Sigma_\ell)$ has the quotient topology. For $s \geq 0$

we may identify $\tilde{H}_0^s(\Sigma_\ell)$ with the subspace of $L_2(\Sigma_\ell)$ which consists of those functions whose extension by 0 yields an element of $H^s(\mathbb{R})$. Furthermore, the space $\tilde{H}_0^s(\Sigma_\ell)$ is an isometric realisation of the (anti-)dual of $H^{-s}(\Sigma_\ell)$ and vice-versa. The dual pairing is given by the expression

$$\langle g, h \rangle_{\Sigma_\ell} := \langle G, h \rangle_{\mathbb{R}}, \quad g \in H^{-s}(\Sigma_\ell), \quad h \in \tilde{H}_0^s(\Sigma_\ell), \quad (2.11)$$

where $G \in H^{-s}(\mathbb{R})$ denotes any extension of g , cf. [25, Theorem 3.14]. Note that $C_c^\infty(\Sigma_\ell)$ is a dense subset of $\tilde{H}_0^{1/2}(\Sigma_\ell)$, cf. [25, Theorem 3.29]. In particular the above expression is independent of the chosen extension G . Thus, the domain of the quadratic form $a_{\ell,b}$ may be rewritten as follows

$$D[a_{\ell,b}] = \left\{ u \in H^1(\Omega) : u|_{\mathbb{R} \times \{\alpha\}} = 0 \wedge u|_{\mathbb{R} \times \{0\}} \in \tilde{H}_0^{1/2}(\Sigma_\ell) \right\}. \quad (2.12)$$

We define the truncated Dirichlet-to-Neumann operator

$$D_{\ell,\omega} : \tilde{H}_0^{1/2}(\Sigma_\ell) \rightarrow H^{-1/2}(\Sigma_\ell), \quad D_{\ell,\omega} := r_\ell D_\omega e_\ell, \quad (2.13)$$

where

$$r_\ell : H^{-1/2}(\mathbb{R}) \rightarrow H^{-1/2}(\Sigma_\ell)$$

is the restriction operator and

$$e_\ell : \tilde{H}_0^{1/2}(\Sigma_\ell) \rightarrow H^{1/2}(\mathbb{R})$$

is the embedding. Identifying $\tilde{H}_0^{1/2}(\Sigma_\ell)$ with a subspace of $L_2(\Sigma_\ell)$, the operator e_ℓ is simply extension by 0. Considering the corresponding topologies one easily observes that $D_{\ell,\omega}$ is a bounded linear operator. Recalling that $b \in L_\infty(\mathbb{R})$, we define the truncated Dirichlet-to-Robin operator by

$$D_{\ell,\omega} + b : \tilde{H}_0^{1/2}(\Sigma_\ell) \rightarrow H^{-1/2}(\Sigma_\ell), \quad D_{\ell,\omega} + b := r_\ell(D_\omega + b)e_\ell, \quad (2.14)$$

where we identify b with the corresponding multiplication operator. The next lemma gives a characterisation of the eigenvalues of $A_{\ell,b}$ in terms of the truncated Dirichlet-to-Robin operator.

Lemma 2.3. *Let $\omega \in \mathbb{C} \setminus [\pi^2/\alpha^2, \infty)$ and $\ell > 0$. Then*

$$\dim \ker(A_{\ell,b} - \omega) = \dim \ker(D_{\ell,\omega} + b).$$

Proof. The assertion follows if we prove that the trace mapping is an isomorphism of $\ker(A_{\ell,b} - \omega)$ onto $\ker(D_{\ell,\omega} + b)$. Let us first prove that it indeed maps $\ker(A_{\ell,b} - \omega)$ into $\ker(D_{\ell,\omega} + b)$. Let $u \in \ker(A_{\ell,b} - \omega)$ and denote $g := u|_{\mathbb{R} \times \{0\}} \in \tilde{H}_0^{1/2}(\Sigma_\ell)$ its boundary trace. Let $h \in \tilde{H}_0^{1/2}(\Sigma_\ell)$ be an arbitrary test function and choose $v \in D[a_{\ell,b}]$, such that $v|_{\mathbb{R} \times \{0\}} = h$. The dual pairing (2.11) and Lemma 2.2 imply

$$\begin{aligned} \langle (D_{\ell,\omega} + b)g, h \rangle_{\Sigma_\ell} &= \langle D_\omega g + bg, h \rangle_{\mathbb{R}} = \langle \nabla u, \nabla v \rangle_\Omega + \langle bg, h \rangle_{\mathbb{R}} - \omega \langle u, v \rangle_\Omega \\ &= a_{\ell,b}[u, v] - \omega \langle u, v \rangle_\Omega = 0, \end{aligned}$$

as $D_\omega g \in H^{1/2}(\mathbb{R})$ is obviously an extension of $D_{\ell,\omega}g \in H^{1/2}(\Sigma_\ell)$ and u is an eigenfunction for the eigenvalue ω . Hence, $g \in \ker(D_{\ell,\omega} + b)$, which proves that the mapping

$$\ker(A_{\ell,b} - \omega) \ni u \mapsto u|_{\mathbb{R} \times \{0\}} \in \ker(D_{\ell,\omega} + b)$$

is well defined. Moreover, Lemma 2.1 implies that this mapping is injective. It remains to prove surjectivity. Let $g \in \ker(D_{\ell,\omega} + b)$ and denote by $u \in H^1(\Omega)$ the unique solution of the Poisson problem (2.3). Then $u \in D[a_{\ell,b}]$ and for arbitrary $v \in D[a_{\ell,b}]$ we have

$$\begin{aligned} a_{\ell,b}[u, v] &= \langle \nabla u, \nabla v \rangle_{\Omega} + \langle bg, v|_{\mathbb{R} \times \{0\}} \rangle_{\mathbb{R}} = \omega \langle u, v \rangle_{\Omega} + \langle D_{\omega}g + bg, v|_{\mathbb{R} \times \{0\}} \rangle_{\mathbb{R}} \\ &= \omega \langle u, v \rangle_{\Omega} + \langle (D_{\ell,\omega} + b)g, v|_{\mathbb{R} \times \{0\}} \rangle_{\Sigma_{\ell}} = \omega \langle u, v \rangle_{\Omega}. \end{aligned}$$

Thus, $u \in D(A_{\ell,b})$ and $(A_{\ell,b} - \omega)u = 0$. This proves the assertion. \square

A particular consequence of Lemma 2.3 is the observation that the Dirichlet-to-Robin operator $D_{\ell,\omega} + b$ has non-trivial kernel if and only if ω is an eigenvalue of $A_{\ell,b}$. Put $V := \tilde{H}_0^{1/2}(\Sigma_{\ell})$ and consider the Gelfand triple

$$V \rightarrow L_2(\Sigma_{\ell}) \rightarrow V^*.$$

We identify V with a subspace of $L_2(\Sigma_{\ell})$ and $V^* = H^{-1/2}(\Sigma_{\ell})$ is the space of antilinear functionals on $\tilde{H}_0^{1/2}(\Sigma_{\ell})$, cf. the dual pairing (2.11). The truncated Dirichlet-to-Robin operator $D_{\ell,\omega}$ maps

$$D_{\ell,\omega} + b : V \rightarrow V^*,$$

and thus, it is completely described by the sesquilinear form

$$(d_{\ell,\omega} + b)[g, h] := \langle (D_{\ell,\omega} + b)g, h \rangle_{V^*, V} = \langle D_{\ell,\omega}g, h \rangle_{\Sigma_{\ell}} + \langle bg, h \rangle_{\Sigma_{\ell}} \quad (2.15)$$

where $g, h \in D[d_{\ell,\omega}] := \tilde{H}_0^{1/2}(\Sigma_{\ell})$. Using the dual pairing (2.11) we obtain

$$d_{\ell,\omega}[g, h] = \langle D_{\omega}g, h \rangle_{\mathbb{R}} = \int_{\mathbb{R}} \sqrt{\xi^2 - \omega} \coth(\alpha \sqrt{\xi^2 - \omega}) \cdot \hat{g}(\xi) \overline{\hat{h}(\xi)} \, d\xi \quad (2.16)$$

and

$$b[g, h] = \int_{\Sigma_{\ell}} b(x) \cdot g(x) \overline{h(x)} \, dx. \quad (2.17)$$

We note that (2.16) is independent of ℓ and the dependence of ℓ in (2.17) is manifested in the domain of integration. In particular we may consider the dependence on ℓ as a constraint on the support of the functions g and h .

Lemma 2.4. *Let $\omega \in \mathbb{C} \setminus [\pi^2/\alpha^2, \infty)$. Then $d_{\ell,\omega}$ defines a closed sectorial form in $L_2(\Sigma_{\ell})$. The associated m -sectorial operator is the restriction of $D_{\ell,\omega} + b$ to the operator domain*

$$X_{\ell,\omega} := \left\{ g \in \tilde{H}_0^{1/2}(\Sigma_{\ell}) : D_{\ell,\omega}g \in L_2(\Sigma_{\ell}) \right\}. \quad (2.18)$$

Proof. Combining Formulae (2.16) and (2.17) we have for $g, h \in \tilde{H}_0^{1/2}(\Sigma_{\ell})$

$$(d_{\ell,\omega} + b)[g, h] = \int_{\mathbb{R}} m_{\omega}(\xi) \cdot \hat{g}(\xi) \overline{\hat{h}(\xi)} \, d\xi + \int_{\Sigma_{\ell}} b(x) \cdot g(x) \overline{h(x)} \, dx$$

with $m_{\omega}(\xi) = \sqrt{\xi^2 - \omega} \coth(\alpha \sqrt{\xi^2 - \omega})$. Note that

$$m_{\omega}(\xi) = \sqrt{\xi^2 - \omega} \coth(\alpha \sqrt{\xi^2 - \omega}) = |\xi| + O(1) \quad \text{as } \xi \rightarrow \pm\infty.$$

As $\tilde{H}_0^{1/2}(\Sigma_\ell)$ carries the subspace topology induced by $H^{1/2}(\mathbb{R})$ this implies

$$c_1^{-1} \|g\|_{\tilde{H}_0^{1/2}(\Sigma_\ell)}^2 \leq \operatorname{Re}(d_{\ell,\omega} + b)[g] + c_2 \|g\|_{L_2(\Sigma_\ell)}^2 \leq c_1 \|g\|_{\tilde{H}_0^{1/2}(\Sigma_\ell)}^2, \quad (2.19)$$

for constants $c_i = c_i(\omega, \alpha, \Sigma_\ell) \in \mathbb{R}$, $i = 1, 2$. Thus, the form $d_{\ell,\omega}$ is bounded from below and closed. Moreover, it easily follows that $d_{\ell,\omega} + b$ is sectorial. To prove the second assertion let $g \in X_{\ell,\omega}$ such that $D_{\ell,\omega}g = \tilde{f} \in L_2(\Sigma_\ell)$. Then,

$$(d_{\ell,\omega} + b)[g, h] = \langle D_{\ell,\omega}g, h \rangle_{\Sigma_\ell} + \langle bg, h \rangle_{\Sigma_\ell} = \langle \tilde{f} + bg, h \rangle_{\Sigma_\ell}$$

for all $h \in C_c^\infty(\Sigma_\ell)$. As $C_c^\infty(\Sigma_\ell)$ is dense in $\tilde{H}_0^{1/2}(\Sigma_\ell)$ the above equality holds true for every $h \in \tilde{H}_0^{1/2}(\Sigma_\ell)$. Thus, g lies in the operator domain of the m-sectorial realisation which acts as $D_{\ell,\omega} + b$. In the same way it follows that the domain of the m-sectorial realisation is contained in $X_{\ell,\omega}$. \square

Since

$$\ker(D_{\ell,\omega} + b) \subseteq X_{\ell,\omega},$$

we can apply Hilbert space methods to determine whether zero is an eigenvalue of $D_{\ell,\omega} + b$ or not. The spectrum of the m-sectorial realisation consists of a discrete set of eigenvalues only accumulating at infinity, since $D[d_{\ell,\omega} + b] = \tilde{H}_0^{1/2}(\Sigma_\ell)$ is compactly embedded into $L_2(\Sigma_\ell)$, cf. [25, Theorem 3.27]. Moreover, for real $\omega \in \mathbb{R} \setminus [\pi^2/\alpha^2, \infty)$ the quadratic form $d_{\ell,\omega} + b$ is symmetric, and thus, the associated operator is self-adjoint.

Remark. The close relation between self-adjoint extensions of differential operators and self-adjoint operators acting on the boundary has been pointed out in the case of bounded domains by G. Grubb, in particular with regard to resolvent formulae, cf. [21] and the references therein.

Another consequence of Lemma 2.4 or merely its proof is the following lemma.

Lemma 2.5. *The (original) operator $D_{\ell,\omega} + b : \tilde{H}_0^{1/2}(\Sigma_\ell) \rightarrow H^{-1/2}(\Sigma_\ell)$ is an Fredholm operator with zero index.*

The proof follows by combining Formula (2.19) and [25, Theorems 2.34 and 3.27].

As the m-sectorial realisation of the Dirichlet-to-Robin operator is simply the restriction of the original operator we do not want to introduce a separate notation for it. In fact, we will mainly work with a quadratic form, which arises after scaling the Robin window. Recall that $\Sigma_\ell := \ell \cdot \Sigma$. We define the unitary scaling operator

$$T_\ell : L_2(\Sigma) \rightarrow L_2(\Sigma_\ell), \quad (T_\ell g)(x) = \ell^{-1/2} g\left(\frac{x}{\ell}\right). \quad (2.20)$$

Note that the operator T_ℓ bijectively maps $\tilde{H}_0^{1/2}(\Sigma)$ into $\tilde{H}_0^{1/2}(\Sigma_\ell)$. Set

$$\mathcal{Q}_b(\ell, \omega) : \tilde{H}_0^{1/2}(\Sigma) \rightarrow H^{-1/2}(\Sigma), \quad \mathcal{Q}_b(\ell, \omega) := T_\ell^*(D_{\ell,\omega} + b)T_\ell \quad (2.21)$$

and let

$$q_b(\ell, \omega)[g, h] := (d_{\ell,\omega} + b)[T_\ell g, T_\ell h] \quad (2.22)$$

be the associated sesquilinear form with $D[q_b(\ell, \omega)] := \tilde{H}_0^{1/2}(\Sigma)$. Then

$$\dim \ker(A_{\ell, b} - \omega) = \dim \ker(D_{\ell, \omega} + b) = \dim \ker(\mathcal{Q}_b(\ell, \omega)).$$

Next we prove an asymptotic expansion of the operator $\mathcal{Q}_b(\ell, \omega)$ as $\ell \rightarrow 0$ and $\omega \rightarrow \pi^2/\alpha^2$. This expansion represents the principal tool of the proof of the main result. Here and subsequently we denote by $\mathcal{Q}_0 : \tilde{H}_0^{1/2}(\Sigma) \rightarrow H^{-1/2}(\Sigma)$,

$$\langle \mathcal{Q}_0 g, h \rangle_\Sigma := q_0[g, h] := \int_{\mathbb{R}} |\xi| \cdot \hat{g}(\xi) \overline{\hat{h}(\xi)} \, d\xi \quad (2.23)$$

the Dirichlet-to-Neumann operator for the mixed problem on the upper half-space or equivalently on the lower half-space corresponding to the spectral parameter $\omega = 0$. Note that $\mathcal{Q}_0 : \tilde{H}_0^{1/2}(\Sigma) \rightarrow H^{-1/2}(\Sigma)$ is also a Fredholm operator with Fredholm index 0, which follows from [25, Theorem 2.34]. The identity $\mathcal{Q}_0 g = 0$ implies

$$0 = \langle \mathcal{Q}_0 g, g \rangle_\Sigma = \int_{\mathbb{R}} |\xi| \cdot |\hat{g}(\xi)|^2 \, d\xi,$$

and thus, $g = 0$. Hence \mathcal{Q}_0 has trivial kernel and it is invertible.

In what follows we denote by P_{ct} the projection in $L_2(\Sigma)$ onto the subspace of constant functions and let $K_{\ln|x|} : L_2(\Sigma) \rightarrow L_2(\Sigma)$,

$$(K_{\ln|x|} f)(z) = \int_{\Sigma} \ln|z - x| \cdot f(x) \, dx, \quad x \in \Sigma. \quad (2.24)$$

Theorem 2.6. *Let $b = 0$. There exist $\ell_0 = \ell_0(\alpha, \Sigma) > 0$ and $\varepsilon = \varepsilon(\alpha, \Sigma) > 0$ such that for $\ell \in (0, \ell_0)$ and $|\omega - \pi^2/\alpha^2| < \varepsilon$ the following asymptotic expansion holds true*

$$\mathcal{Q}_0(\ell, \omega) = \frac{1}{\ell} \mathcal{Q}_0 - \ell \cdot \left(\frac{|\Sigma| \cdot \pi^2}{\alpha^3} \right) \cdot \frac{1}{\sqrt{\pi^2/\alpha^2 - \omega}} P_{\text{ct}} \quad (2.25)$$

$$+ \sum_{k_1=1}^{\infty} \sum_{k_2=0}^{\infty} \ell^{2k_1-1} \left(\sqrt{\pi^2/\alpha^2 - \omega} \right)^{k_2} \left(B_{k_1, k_2}^{(0)} + B_{k_1, k_2}^{(1)} \cdot \ln \ell \right). \quad (2.26)$$

Here $B_{k_1, k_2}^{(i)} \in \mathcal{L}(L_2(\Sigma))$ for $i = 1, 2$. The series converges absolutely in the operator norm of $\mathcal{L}(L_2(\Sigma))$. For the first terms we obtain

$$B_{1,0}^{(0)} = \frac{|\Sigma| \cdot \rho(\alpha)}{2\pi} P_{\text{ct}} + \frac{\pi}{2\alpha^2} K_{\ln|x|}, \quad B_{1,0}^{(1)} = \frac{|\Sigma| \cdot \pi}{2\alpha^2} P_{\text{ct}}, \quad B_{1,1}^{(0)} = B_{1,1}^{(1)} = 0,$$

where the constant $\rho(\alpha) \in \mathbb{R}$ is given by Formula (2.44) and $|\Sigma|$ is the Lebesgue measure of Σ .

The next section is devoted to the proof of Theorem 2.6.

The proof of Theorem 2.6. For $g, h \in \tilde{H}_0^{1/2}(\Sigma)$ we have

$$\langle \mathcal{Q}_0(\ell, \omega) g, h \rangle_\Sigma = q_0(\ell, \omega)[g, h] = \ell \int_{\mathbb{R}} m_\omega(\xi) \cdot \hat{g}(\ell\xi) \overline{\hat{h}(\ell\xi)} \, d\xi,$$

where $m_\omega(\xi) = \sqrt{\xi^2 - \omega} \cdot \coth(\alpha\sqrt{\xi^2 - \omega})$. The main idea of the proof is to use the asymptotic expansion of the function $m_\omega(\xi)$ for $\xi = 0$ and $\xi \rightarrow \pm\infty$ while letting the parameter $\omega \rightarrow \pi^2/\alpha^2$.

As a first step of the proof we show that m_ω has a meromorphic extension to the complex plane and calculate explicitly its singularities and residues. To this end we use the partial fraction decomposition of the hyperbolic cotangent function, i.e., we have

$$\coth(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{2z}{z^2 + k^2\pi^2}, \quad z \in \mathbb{C} \setminus \{ik\pi : k \in \mathbb{Z}\}, \quad (2.27)$$

cf. e.g. [24, Chapter V, §1.71]. Hence,

$$m_\omega(\xi) = \frac{1}{\alpha} + \sum_{k=1}^{\infty} \frac{2\alpha(\xi^2 - \omega)}{\alpha^2(\xi^2 - \omega) + k^2\pi^2}, \quad \xi \in \mathbb{R}, \quad (2.28)$$

and the meromorphic extension of m_ω is given by the series (2.28). For $\omega \in \mathbb{C} \setminus [\pi^2/\alpha^2, \infty)$ the singularities of m_ω are all simple poles which are located at

$$\pm i\sqrt{\frac{k^2\pi^2}{\alpha^2} - \omega}, \quad k \in \mathbb{N}. \quad (2.29)$$

In particular they do not lie on the real axis. As $\omega \rightarrow \pi^2/\alpha^2$ the two poles nearest the real axis converge to $0 \in \mathbb{C}$ and they give rise to a pole of order two in the limit case. Here and subsequently we fix $\beta = \beta(\alpha) \in (\pi/\alpha, \sqrt{3}\pi/\alpha)$. Then there exists $\varepsilon = \varepsilon(\alpha) > 0$ such that for $\omega \in \mathbb{C} \setminus [\pi^2/\alpha^2, \infty)$, $|\omega - \pi^2/\alpha^2| < \varepsilon$ the function m_ω has exactly two poles inside the strip $\mathbb{R} + i[-\beta, \beta]$. The residues of the function m_ω at these points are given by

$$\text{Res}_{\xi=\pm i\sqrt{\pi^2/\alpha^2-\omega}} m_\omega(\xi) = \pm \frac{\pi^2}{\alpha^3} \cdot \frac{i}{\sqrt{\pi^2/\alpha^2-\omega}} \quad (2.30)$$

as can easily be seen from the expansion (2.28).

Let $g, h \in \tilde{H}_0^{1/2}(\Sigma)$. Since g, h are compactly supported their Fourier transforms \hat{g}, \hat{h} admit holomorphic extensions on the whole complex plane. Note that the function \hat{h}^* ,

$$\hat{h}^*(\xi) := \overline{\hat{h}(\bar{\xi})}, \quad \xi \in \mathbb{C},$$

is also an entire function on \mathbb{C} . We decompose the form $q_0(\ell, \omega)$ as follows

$$q_0(\ell, \omega)[g, h] = \ell \left(\int_{[-1,1]} + \int_{[-1,1]^c} \right) m_\omega(\xi) \cdot \hat{g}(\ell\xi) \hat{h}^*(\ell\xi) \, d\xi, \quad (2.31)$$

where we put $[-1,1]^c = \mathbb{R} \setminus [-1,1]$. Using the Taylor expansion of the function $\hat{g} \cdot \hat{h}^*$ at $0 \in \mathbb{C}$, we obtain for the first integral

$$\int_{[-1,1]} m_\omega(\xi) \cdot \hat{g}(\ell\xi) \hat{h}^*(\ell\xi) \, d\xi = \sum_{k=0}^{\infty} \ell^k e_k[g, h] \int_{[-1,1]} \xi^k m_\omega(\xi) \, d\xi \quad (2.32)$$

with

$$e_k[g, h] = \frac{1}{k!} \cdot \frac{d^k}{d\xi^k} (\hat{g}(\xi) \hat{h}^*(\xi)) \Big|_{\xi=0} = \frac{1}{k!} \cdot \sum_{j=0}^k \binom{k}{j} \hat{g}^{(j)}(0) \cdot \overline{\hat{h}^{(k-j)}(0)}. \quad (2.33)$$

We note that m_ω is an even function, and thus, in the expansion the terms of odd order vanish. Let E_k be the operator associated with the form e_k . Then

$$\int_{[-1,1]} m_\omega(\xi) \cdot \hat{g}(\ell\xi) \hat{h}^*(\ell\xi) \, d\xi = \sum_{k=0}^{\infty} \ell^{2k} \langle E_{2k} g, h \rangle_\Sigma \int_{[-1,1]} \xi^{2k} m_\omega(\xi) \, d\xi.$$

Note that

$$|\hat{g}^{(j)}(0)| \leq \frac{1}{\sqrt{2\pi}} \left(\int_{\Sigma} |x|^{2j} \right)^{1/2} \|g\|_{L_2(\Sigma)} \leq C^j \|g\|_{L_2(\Sigma)}$$

for sufficiently large $C = C(\Sigma) > 0$, which implies

$$\|E_k\|_{\mathcal{L}(L_2(\Sigma))} \leq \frac{(2C)^k}{k!}.$$

To estimate the integral $\int_{[-1,1]} \xi^{2k} m_\omega(\xi) \, d\xi$ we denote by γ the path depicted in Figure 4 connecting the points -1 and 1 . Its image $\text{im}(\gamma)$ coincides with the boundary of the following rectangle except for the line segment $[-1, 1]$.

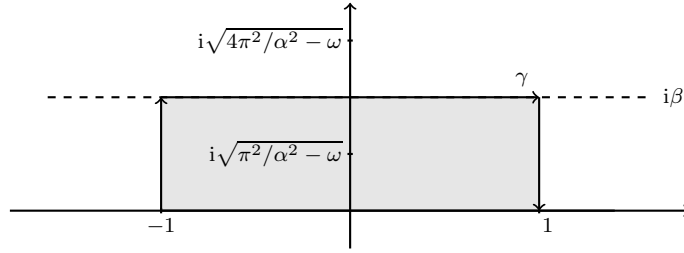


FIGURE 4. The path γ .

Note that $\text{Im}(i\sqrt{\pi^2/\alpha^2 - \omega}) > 0$ if $\omega \in \mathbb{C} \setminus [\pi^2/\alpha^2, \infty)$. Using Formula (2.30) the residue theorem implies

$$\int_{[-1,1]} \xi^{2k} m_\omega(\xi) \, d\xi = -\frac{(-1)^k \cdot 2\pi^3}{\alpha^3} (\pi^2/\alpha^2 - \omega)^{k-1/2} + \int_{\gamma} \xi^{2k} m_\omega(\xi) \, d\xi. \quad (2.34)$$

Next we use for fixed $\xi \in \text{im}(\gamma)$ the Taylor expansion of $m_\omega(\xi)$ at $\omega = \pi^2/\alpha^2$. Thus, there exists $\varepsilon > 0$ such that for $|\omega - \pi^2/\alpha^2| < \varepsilon$ we have

$$m_\omega(\xi) = \sum_{j=0}^{\infty} \frac{(-1)^j (\pi^2/\alpha^2 - \omega)^j}{j!} \left. \frac{d^j}{d\omega^j} m_\omega(\xi) \right|_{\omega=\pi^2/\alpha^2}. \quad (2.35)$$

This expression may be considered as a power series in ω with values in $L_1(\text{im}(\gamma))$, and we obtain

$$\int_{\gamma} \xi^{2k} m_\omega(\xi) \, d\xi = \sum_{j=0}^{\infty} \frac{(-1)^j (\pi^2/\alpha^2 - \omega)^j}{j!} \left[\int_{\gamma} \xi^{2k} \frac{d^j}{d\omega^j} m_\omega(\xi) \, d\xi \right]_{\omega=\pi^2/\alpha^2}.$$

Finally,

$$\begin{aligned}
& \ell \int_{[-1,1]} m_\omega(\xi) \cdot \hat{g}(\ell\xi) \hat{h}^*(\ell\xi) \, d\xi \\
&= -\ell \left(\frac{2\pi^3}{\alpha^3} \right) \cdot \frac{1}{\sqrt{\pi^2/\alpha^2 - \omega}} \langle E_0 g, h \rangle_\Sigma \\
&\quad - \left(\frac{2\pi^3}{\alpha^3} \right) \cdot \sum_{k=1}^{\infty} \ell^{2k+1} \cdot (-1)^k \cdot (\pi^2/\alpha^2 - \omega)^{k-1/2} \cdot \langle E_{2k} g, h \rangle_\Sigma \\
&\quad + \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \ell^{2k+1} \langle E_{2k} g, h \rangle_\Sigma \frac{(-1)^j (\pi^2/\alpha^2 - \omega)^j}{j!} \left[\int_\gamma \xi^{2k} \frac{d^j}{d\omega^j} m_\omega(\xi) \, d\xi \right]_{\omega=\pi^2/\alpha^2}.
\end{aligned}$$

We note that the two series

$$\sum_{k=1}^{\infty} \ell^{2k+1} E_{2k} \cdot (-1)^k \cdot (\pi^2/\alpha^2 - \omega)^{k-1/2}$$

and

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \ell^{2k+1} E_{2k} \frac{(-1)^j (\pi^2/\alpha^2 - \omega)^j}{j!} \left[\int_\gamma \xi^{2k} \frac{d^j}{d\omega^j} m_\omega(\xi) \, d\xi \right]_{\omega=\pi^2/\alpha^2}$$

converge absolutely in the operator norm in $\mathcal{L}(L_2(\Sigma))$, uniformly in $\ell \in [0, 1]$ and $|\omega - \pi^2/\alpha^2| < \varepsilon$. For the first series this is obvious. Considering the second series leads us to the estimate

$$\begin{aligned}
& \sum_{k=0}^{\infty} \ell^{2k+1} \|E_{2k}\|_{\mathcal{L}(L_2(\Sigma))} \sum_{j=0}^{\infty} \frac{|\pi^2/\alpha^2 - \omega|^j}{j!} \left| \int_\gamma \xi^{2k} \frac{d^j}{d\omega^j} m_{\pi^2/\alpha^2}(\xi) \, d\xi \right| \\
& \leq \left(\sum_{k=0}^{\infty} \ell^{2k+1} \cdot c_1^{2k} \cdot \frac{(2C)^{2k}}{(2k)!} \right) \left(\sum_{j=0}^{\infty} \frac{\varepsilon^j}{j!} \left\| \frac{d^j}{d\omega^j} m_{\pi^2/\alpha^2} \right\|_{L_1(\text{im}(\gamma))} \right) < \infty
\end{aligned}$$

for $\ell > 0$ and $|\omega - \pi^2/\alpha^2| < \varepsilon$. Here $c_1 := \sup\{|\xi| : \xi \in \text{im}(\gamma)\}$. Calculating the first terms of the expansion we obtain

$$\begin{aligned}
& \ell \int_{[-1,1]} m_\omega(\xi) \cdot \hat{g}(\ell\xi) \hat{h}^*(\ell\xi) \, d\xi \\
&= -\ell \cdot \frac{2\pi^3 (\hat{g}\hat{h}^*)(0)}{\alpha^3 \sqrt{\pi^2/\alpha^2 - \omega}} + \ell \cdot (\hat{g}\hat{h}^*)(0) \int_\gamma m_{\pi^2/\alpha^2}(\xi) \, d\xi + \mathcal{O}(\ell^3 + \pi^2/\alpha^2 - \omega).
\end{aligned}$$

Let P_{ct} be the projection in $L_2(\Sigma)$ onto the subspace of constant functions. Then

$$(\hat{g}\hat{h}^*)(0) = \frac{1}{2\pi} \left(\int_\Sigma g(x) \, dx \right) \left(\int_\Sigma \overline{h(x)} \, dx \right) = \frac{|\Sigma|}{2\pi} \langle P_{\text{ct}} g, h \rangle_\Sigma, \quad (2.36)$$

and thus,

$$\begin{aligned} & \ell \int_{[-1,1]} m_\omega(\xi) \cdot \hat{g}(\ell\xi) \hat{h}^*(\ell\xi) \, d\xi \\ &= \ell \left(-\frac{|\Sigma| \cdot \pi^2}{\alpha^3 \sqrt{\pi^2/\alpha^2 - \omega}} + \frac{|\Sigma| \cdot \rho_{0,1}(\alpha)}{2\pi} \right) \langle P_{\text{ct}}g, h \rangle_\Sigma + \mathcal{O}(\ell^3 + \pi^2/\alpha^2 - \omega), \end{aligned}$$

with $\rho_{0,1}(\alpha) = \int_\delta m_{\pi^2/\alpha^2}(\xi) \, d\xi$. In order to treat the second integral in (2.31) we use the asymptotic expansion of $m_\omega(\xi)$ for $\xi \rightarrow \pm\infty$. For ease of notation we suppose that $\alpha > \pi$, so that $[-1, 1] \subseteq (-\pi/\alpha, \pi/\alpha)$. We have

$$\begin{aligned} & \ell \int_{[-1,1]^c} m_\omega(\xi) \cdot \hat{g}(\ell\xi) \hat{h}^*(\ell\xi) \, d\xi \\ &= \ell \int_{\mathbb{R}} |\xi| \cdot \hat{g}(\ell\xi) \hat{h}^*(\ell\xi) \, d\xi + \ell \sum_{i=1}^3 \int_{\mathbb{R}} m_{\omega,i}(\xi) \cdot \hat{g}(\ell\xi) \hat{h}^*(\ell\xi) \, d\xi, \end{aligned}$$

where

$$m_{\omega,1}(\xi) := -\mathbb{1}_{[-1,1]}(\xi) \cdot |\xi|, \quad (2.37)$$

$$m_{\omega,2}(\xi) := \mathbb{1}_{[-1,1]^c}(\xi) \cdot \left(\sqrt{\xi^2 - \omega} - |\xi| \right), \quad (2.38)$$

$$m_{\omega,3}(\xi) := \mathbb{1}_{[-1,1]^c}(\xi) \sqrt{\xi^2 - \omega} \left(\coth(\alpha \sqrt{\xi^2 - \omega}) - 1 \right). \quad (2.39)$$

Here $\mathbb{1}_E$ denotes the indicator function of a Borel set $E \subseteq \mathbb{R}$. We note that the first term $\int_{\mathbb{R}} m_{\omega,1}(\xi) \cdot \hat{g}(\ell\xi) \hat{h}^*(\ell\xi) \, d\xi$ is independent of ω and may be expanded as before into a power series with respect to the parameter ℓ ; we have

$$\begin{aligned} \ell \int_{\mathbb{R}} m_{\omega,1}(\xi) \cdot \hat{g}(\ell\xi) \hat{h}^*(\ell\xi) \, d\xi &= \ell \int_{[-1,1]} |\xi| \cdot \hat{g}(\ell\xi) \hat{h}^*(\ell\xi) \, d\xi \\ &= \frac{\ell|\Sigma|}{2\pi} \langle P_{\text{ct}}g, h \rangle_\Sigma \cdot \int_{[-1,1]} |\xi| \, d\xi + \mathcal{O}(\ell^3) \\ &= \frac{\ell|\Sigma|}{2\pi} \langle P_{\text{ct}}g, h \rangle + \mathcal{O}(\ell^3). \end{aligned}$$

To treat the second integral we use for $\xi \in \mathbb{R}$ and $|\omega - \pi^2/\alpha^2| < \varepsilon$ the following expansion

$$\begin{aligned} m_{\omega,2}(\xi) &= \mathbb{1}_{[-1,1]^c}(\xi) \left(\sqrt{\xi^2 - \omega} - |\xi| \right) = \mathbb{1}_{[-1,1]^c}(\xi) \cdot \left(|\xi| \cdot \sqrt{1 - \frac{\omega}{\xi^2}} - |\xi| \right) \\ &= \sum_{k=1}^{\infty} \binom{1/2}{k} (-\omega)^k \cdot |\xi|^{-2k+1} \mathbb{1}_{[-1,1]^c}(\xi). \end{aligned}$$

This series may be considered as power series in ω with values in $L_\infty(\mathbb{R}_\xi)$. Let $Y_k, Z_k \in C^\infty(\mathbb{R} \setminus \{0\}) \cap L_{1,\text{loc}}(\mathbb{R})$ such that

$$\hat{Y}_k(\xi) = \frac{1}{\sqrt{2\pi}} \cdot |\xi|^{-2k+1} \mathbb{1}_{[-1,1]^c}(\xi) \quad \text{and} \quad \hat{Z}_k(\xi) = \frac{1}{\sqrt{2\pi}} \, \text{f.p.} \left(|\xi|^{-2k+1} \right).$$

Here

$$\text{f.p.} \left(|\xi|^{-2k+1} \right) = \text{f.p.} \left(\xi_+^{-2k+1} \right) + \text{f.p.} \left(\xi_-^{-2k+1} \right)$$

designates the distribution which is defined by the finite part of the singular function $|\xi|^{-2k+1}$, cf. [25, Chapter 5]. We note that $X_k := Y_k - Z_k$ is analytic, since its Fourier transform $\hat{X}_k = \hat{Y}_k - \hat{Z}_k$ has compact support. This allows us to determine the order of the singularity of Y_k at $0 \in \mathbb{R}$. Using [25, Lemma 5.10] we have

$$Z_k(x) = \frac{1}{\pi} \cdot \frac{(-1)^k x^{2k-2}}{(2k-2)!} (\ln|x| + \gamma_0 - H_{2k-2}), \quad (2.40)$$

where γ_0 is the Euler-Mascheroni constant and $H_{2k-2} = \sum_{j=1}^{2k-2} \frac{1}{j}$. We have

$$\begin{aligned} \ell \int_{\mathbb{R}} m_{\omega,2}(\xi) \cdot \hat{g}(\ell\xi) \hat{h}^*(\ell\xi) \, d\xi &= \sum_{k=1}^{\infty} \binom{1/2}{k} (-\omega)^k \cdot \ell \cdot \int_{\mathbb{R}} \hat{Y}_k(\xi) \cdot \hat{g}(\ell\xi) \hat{h}^*(\ell\xi) \, d\xi \\ &= \sum_{k=1}^{\infty} \binom{1/2}{k} (-\omega)^k \cdot \langle Y_k * T_{\ell}g, T_{\ell}h \rangle_{\Sigma_{\ell}}, \end{aligned}$$

$$\langle Y_k * T_{\ell}g, T_{\ell}h \rangle_{\Sigma_{\ell}} = \ell \int_{\Sigma \times \Sigma} (X_k + Z_k)(\ell|z-x|) g(x) \overline{h(z)} \, dx \, dz.$$

Defining the operators $K_{|x|^{2k-2}}, K_{|x|^{2k-2} \ln|x|} : L_2(\Sigma) \rightarrow L_2(\Sigma)$,

$$(K_{|x|^{2k-2}}f)(z) = \int_{\Sigma} |z-x|^{2k-2} f(x) \, dx, \quad z \in \Sigma \quad (2.41)$$

$$(K_{|x|^{2k-2} \ln|x|}f)(z) = \int_{\Sigma} |z-x|^{2k-2} \ln|z-x| f(x) \, dx, \quad z \in \Sigma \quad (2.42)$$

we obtain

$$\begin{aligned} &\ell \int_{\Sigma \times \Sigma} Z_k(\ell|z-x|) g(x) \overline{h(z)} \, dx \, dz \\ &= \frac{(-1)^k \ell^{2k-1}}{\pi \cdot (2k-2)!} \int_{\Sigma \times \Sigma} |z-x|^{2k-2} \ln(\ell|z-x|) g(x) \overline{h(z)} \, dx \, dz \\ &\quad + \frac{(-1)^k \ell^{2k-1}}{\pi \cdot (2k-2)!} \cdot (\gamma_0 - H_{2k-2}) \int_{\Sigma \times \Sigma} |z-x|^{2k-2} g(x) \overline{h(z)} \, dx \, dz \\ &= \frac{(-1)^k \ell^{2k-1}}{\pi \cdot (2k-2)!} (\ln \ell + \gamma_0 - H_{2k-2}) \langle K_{|x|^{2k-2}}g, h \rangle_{\Sigma} \\ &\quad + \frac{(-1)^k \ell^{2k-1}}{\pi \cdot (2k-2)!} \langle K_{|x|^{2k-2} \ln|x|}g, h \rangle_{\Sigma}. \end{aligned}$$

Note that

$$\ell \int_{\Sigma \times \Sigma} X_k(\ell|z-x|) g(x) \overline{h(z)} \, dx \, dz = \sum_{j=0}^{\infty} \frac{X_k^{(2j)}(0) \cdot \ell^{2j+1}}{(2j)!} \langle K_{|x|^{2j}}g, h \rangle_{\Sigma},$$

which follows by expanding the even function X_k into a power series. For the coefficients $X_k^{(2j)}(0)$ we obtain from the definition of the finite part

$$\begin{aligned} 2\pi X_k^{(2j)}(0) &= 2 \int_0^\infty (i\xi)^{2j} (\hat{Y}_k - \hat{Z}_k)(\xi) \, d\xi = -2(-1)^j \text{f.p.}_{\varepsilon \rightarrow 0} \int_\varepsilon^1 \xi^{2j} \cdot \xi^{-2k+1} \, d\xi \\ &= (-1)^{j+1} \cdot \begin{cases} \frac{1}{j-k+1}, & j-k+1 \neq 0, \\ 0, & j-k+1 = 0. \end{cases} \end{aligned}$$

Finally, we have

$$\begin{aligned} &\ell \int_{\mathbb{R}} m_{\omega,2}(\xi) \cdot \hat{g}(\ell\xi) \hat{h}^*(\ell\xi) \, d\xi \\ &= \sum_{k=1}^\infty \binom{1/2}{k} \frac{\omega^k \cdot \ell^{2k-1}}{\pi(2k-2)!} \left[(\ln \ell + \gamma_0 - H_{2k-2}) \langle K_{|x|^{2k-2}} g, h \rangle_\Sigma + \langle K_{|x|^{2k-2} \ln |x|} g, h \rangle_\Sigma \right] \\ &\quad + \sum_{k=1}^\infty \sum_{j=0}^\infty \binom{1/2}{k} \cdot (-\omega)^k \cdot \frac{X_k^{(2j)}(0) \cdot \ell^{2j+1}}{(2j)!} \langle K_{|x|^{2j}} g, h \rangle_\Sigma. \end{aligned}$$

Note that both series,

$$\frac{1}{\pi} \sum_{k=1}^\infty \binom{1/2}{k} \cdot \omega^k \cdot \frac{\ell^{2k-1}}{(2k-2)!} \left[(\ln \ell + \gamma_0 - H_{2k-2}) K_{|x|^{2k-2}} + K_{|x|^{2k-2} \ln |x|} \right]$$

and

$$\sum_{k=1}^\infty \sum_{j=0}^\infty \binom{1/2}{k} (-\omega)^k \cdot \frac{X_k^{(2j)}(0) \cdot \ell^{2j+1}}{(2j)!} K_{|x|^{2j}},$$

converge uniformly in the operator norm for $\ell \in (0, \ell_0)$ and $|\omega - \pi^2/\alpha^2| < \varepsilon$. This follows from the estimates on the coefficients $X_k^{(2j)}(0)$ and from

$$\|K_{|x|^{2k-2}}\|_{\mathcal{L}(L_2(\Sigma))} \leq C^{2k-2}, \quad \|K_{|x|^{2k-2} \ln |x|}\|_{\mathcal{L}(L_2(\Sigma))} \leq C^{2k-2}$$

for sufficiently large $C = C(\Sigma) > 0$. Note that $\omega < 1$ since we assumed that $\pi^2/\alpha^2 < 1$. Changing the centre of the power series in ω and calculating the first terms give us the following asymptotic estimate

$$\begin{aligned} &\ell \int_{\mathbb{R}} m_{\omega,2}(\xi) \cdot \hat{g}(\ell\xi) \hat{h}^*(\ell\xi) \, d\xi \\ &= \frac{\ell \ln \ell \cdot |\Sigma| \cdot \pi}{2\alpha^2} \langle P_{\text{ct}} g, h \rangle_\Sigma + \ell \cdot |\Sigma| \cdot \left(\frac{\pi \cdot \gamma_0}{2\alpha^2} + \frac{\rho_{0,2}(\alpha)}{2\pi} \right) \langle P_{\text{ct}} g, h \rangle_\Sigma \\ &\quad + \ell \cdot \frac{\pi}{2\alpha^2} \langle K_{\ln |x|} g, h \rangle_\Sigma + \mathcal{O}(\ell^3 \ln \ell + \pi^2/\alpha^2 - \omega), \end{aligned}$$

where

$$\rho_{0,2}(\alpha) := \pi \sum_{k=1}^\infty \binom{1/2}{k} \left(-\frac{\pi^2}{\alpha^2} \right)^k X_k(0) = \frac{1}{2} \sum_{k=2}^\infty \binom{1/2}{k} \left(-\frac{\pi^2}{\alpha^2} \right)^k \frac{1}{k-1}. \quad (2.43)$$

We note that $K_{|x|^0} = |\Sigma| \cdot P_{\text{ct}}$. Thus, the only point remaining is the expansion of the integral $\ell^2 \int_{\mathbb{R}} m_{\omega,3}(\xi) \cdot \hat{g}(\ell\xi) \hat{h}^*(\ell\xi) d\xi$. We have

$$m_{\omega,3}(\xi) = \mathbb{1}_{[-1,1]^c}(\xi) \sqrt{\xi^2 - \omega} \left(\coth(\alpha \sqrt{\xi^2 - \omega}) - 1 \right).$$

It easily follows that the function

$$\omega \mapsto e^{\delta|\xi|} m_{\omega,3}(\xi) \in L_{\infty}(\mathbb{R}_{\xi})$$

is a vector-valued holomorphic function for $|\omega - \pi^2/\alpha^2| < \varepsilon$ and some $\delta > 0$. In particular, we obtain that

$$m_{\omega,3}(\xi) = \sum_{k=0}^{\infty} \frac{(-1)^k (\pi^2/\alpha^2 - \omega)^k}{k!} \frac{d^k}{d\omega^k} m_{\omega,3}(\xi) \Big|_{\omega=\pi^2/\alpha^2}$$

and the series converges absolutely as a power series in ω with values in some exponentially weighted L_{∞} -space. Choose $\tilde{X}_k \in C^{\infty}(\mathbb{R})$ such that

$$\widehat{\tilde{X}_k}(\xi) = \frac{1}{\sqrt{2\pi}} \cdot \frac{d^k}{d\omega^k} m_{\omega,3}(\xi) \Big|_{\omega=\pi^2/\alpha^2}.$$

Then \tilde{X} is an even function and analytic in some neighbourhood of 0,

$$\tilde{X}_k(x) = \sum_{j=0}^{\infty} \frac{\tilde{X}_k^{(2j)}(0)}{(2j)!} x^{2j},$$

where $\tilde{X}_k^{(2j)}(0) = \frac{1}{2\pi} \int_{\mathbb{R}} (i\xi)^{2j} \frac{d^k}{d\omega^k} m_{\pi^2/\alpha^2,3}(\xi) d\xi$. Note that

$$\begin{aligned} |\tilde{X}_k^{(2j)}(0)| &\leq \frac{1}{2\pi} \left\| e^{\delta|\xi|} \frac{d^k}{d\omega^k} m_{\pi^2/\alpha^2,3}(\xi) \right\|_{L_{\infty}(\mathbb{R}_{\xi})} 2 \int_0^{\infty} \xi^{2j} e^{-\delta\xi} d\xi \\ &= \frac{\delta^{-1-2j} (2j)!}{\pi} \left\| e^{\delta|\xi|} \frac{d^k}{d\omega^k} m_{\pi^2/\alpha^2,3}(\xi) \right\|_{L_{\infty}(\mathbb{R}_{\xi})}, \end{aligned}$$

since $\int_0^{\infty} \xi^{2j} e^{-\delta\xi} d\xi = \delta^{-1-2j} (2j)!$, cf. [20, Formula 2.321]. In particular, we obtain

$$\begin{aligned} &\ell \int_{\mathbb{R}} m_{\omega,3}(\xi) \cdot \hat{g}(\ell\xi) \hat{h}^*(\ell\xi) d\xi \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k (\pi^2/\alpha^2 - \omega)^k}{k!} \langle \tilde{X}_k * T_{\ell} g, T_{\ell} h \rangle_{\Sigma_{\ell}} \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^k (\pi^2/\alpha^2 - \omega)^k}{k!} \frac{\tilde{X}_k^{(2j)}(0)}{(2j)!} \cdot \ell^{2j+1} \cdot \langle K_{|x|^{2j}} g, h \rangle_{\Sigma}. \end{aligned}$$

Note that the estimates on the coefficients $\tilde{X}_k^{(2j)}(0)$ imply that the series

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^k (\pi^2/\alpha^2 - \omega)^k}{k!} \frac{\tilde{X}_k^{(2j)}(0)}{(2j)!} \cdot \ell^{2j+1} \cdot K_{|x|^{2j}},$$

converges in the operator norm of $\mathcal{L}(L_2(\Sigma))$, uniformly in $\ell \in [0, \ell_0]$ and $|\omega - \pi^2/\alpha^2| < \varepsilon$. In particular, we obtain

$$\begin{aligned} & \ell \int_{\mathbb{R}} m_{\omega,3}(\xi) \cdot \hat{g}(\ell\xi) \hat{h}^*(\ell\xi) \, d\xi \\ &= \frac{\ell \cdot |\Sigma| \cdot \langle P_{\text{ct}}g, h \rangle_{\Sigma}}{2\pi} \int_{\mathbb{R}} m_{\pi^2/\alpha^2,3}(\xi) \, d\xi + \mathcal{O}(\ell^3 + \pi^2/\alpha^2 - \omega) \\ &= \frac{\ell \cdot |\Sigma| \cdot \rho_{0,3}(\alpha)}{2\pi} \langle P_{\text{ct}}g, h \rangle_{\Sigma} + \mathcal{O}(\ell^3 + \pi^2/\alpha^2 - \omega), \end{aligned}$$

where $\rho_{0,3}(\alpha) = \int_{\mathbb{R}} m_{\pi^2/\alpha^2,3}(\xi) \, d\xi$. Putting

$$\rho_0(\alpha) = \rho_{0,1}(\alpha) + \rho_{0,2}(\alpha) + \rho_{0,3}(\alpha) + 1 + \frac{\gamma_0 \pi^2}{\alpha^2}, \quad (2.44)$$

with

$$\begin{aligned} \rho_{0,1}(\alpha) &= \int_{\delta} m_{\pi^2/\alpha^2}(\xi) \, d\xi, & \rho_{0,2}(\alpha) &= \frac{1}{2} \sum_{k=2}^{\infty} \binom{1/2}{k} \left(-\frac{\pi^2}{\alpha^2}\right)^k \frac{1}{k-1}, \\ \rho_{0,3}(\alpha) &= \int_{\mathbb{R}} m_{\pi^2/\alpha^2,3}(\xi) \, d\xi. \end{aligned}$$

proves Theorem 2.6.

The asymptotic behaviour of the ground state eigenvalue of $A_{\ell,b}$. Recall that $b \in L_{\infty}(\mathbb{R})$ and $\Sigma_{\ell} = \ell \cdot \Sigma \subseteq \mathbb{R}$, where $\Sigma \subseteq \mathbb{R}$ is a finite union of bounded intervals. The following theorems provide the asymptotic behaviour of the ground state eigenvalue as the window length decreases.

Theorem 2.7. *There exists $\ell_0 = \ell_0(\alpha, b, \Sigma) > 0$ such that for all $\ell \in (0, \ell_0)$ the operator $A_{\ell,b}$ has a unique eigenvalue $\lambda(\ell)$ below its essential spectrum. It satisfies*

$$\sqrt{\pi^2/\alpha^2 - \lambda(\ell)} = \ell^2 \left(\frac{\pi^2}{\alpha^3} \right) \cdot \tau_0(\Sigma) + \mathcal{O}(\ell^3) \quad \text{as } \ell \rightarrow 0. \quad (2.45)$$

The constant $\tau_0(\Sigma) > 0$ is given by (2.53). If b is continuously differentiable in some neighbourhood of 0, then the next term of the asymptotic formula is given by

$$\ell^3 \left(\frac{b(0) \cdot \tau_1(\Sigma) \cdot \pi^2}{\alpha^3} \right) \quad (2.46)$$

up to an error of order $\mathcal{O}(\ell^4 \cdot \ln \ell)$. The constant $\tau_1(\Sigma) > 0$ is given by (2.54).

For the special case $\Sigma = (-1, 1)$ we obtain:

Theorem 2.8. *Let $\Sigma_\ell = (-\ell, \ell)$ and let b be twice differentiable in some neighbourhood of 0. Then the eigenvalue $\lambda(\ell)$ satisfies*

$$\begin{aligned} \sqrt{\pi^2/\alpha^2 - \lambda(\ell)} &= \ell^2 \left(\frac{\pi^3}{2\alpha^3} \right) + \ell^3 \left(\frac{4b(0)\pi^2}{3\alpha^3} \right) - \ell^4 \ln \ell \left(\frac{\pi^5}{8\alpha^5} \right) \\ &\quad + \ell^4 \left(\frac{\rho_0(\alpha)\pi^3}{8\alpha^3} + \frac{\pi^5}{32\alpha^5} (1 + \ln 16) - b(0)^2 \cdot \frac{\rho_1 \cdot \pi^2}{\alpha^3} \right) + \mathcal{O}(\ell^5 \ln \ell) \end{aligned} \quad (2.47)$$

as $\ell \rightarrow 0$. The constant $\rho_1 > 0$ is given by (2.56).

First we prove the existence and the uniqueness of the eigenvalue of the operator $A_{\ell,b}$ for small $\ell > 0$. To this end we use the asymptotic expansion in Theorem 2.6 in its weaker form

$$\mathcal{Q}_b(\ell, \omega) = \frac{1}{\ell} \mathcal{Q}_0 - \ell \cdot \left(\frac{|\Sigma| \cdot \pi^2}{\alpha^3} \right) \cdot \frac{1}{\sqrt{\pi^2/\alpha^2 - \omega}} P_{\text{ct}} + R_b(\ell, \omega) \quad (2.48)$$

with the following estimate on the remainder

$$\sup\{\|R_b(\ell, \omega)\|_{\mathcal{L}(L_2(\Sigma))} : \ell \in (0, \ell_0) \wedge |\omega - \pi^2/\alpha^2| < \varepsilon\} < \infty. \quad (2.49)$$

We note that $|\langle bT_\ell g, T_\ell h \rangle_{\Sigma_\ell}| \leq \ell \|b\|_{L_\infty(\Sigma)} \|g\|_{L_2(\Sigma)} \|h\|_{L_2(\Sigma)}$.

Remark. Using a similar argumentation as in Theorem 2.6 it follows that for every compact set $K \subseteq \mathbb{C} \setminus [\pi^2/\alpha^2, \infty)$ there exists $\ell_0 = \ell_0(\alpha, b, \Sigma, K)$ such that

$$\mathcal{Q}_b(\ell, \omega) = \frac{1}{\ell} \mathcal{Q}_0 + \tilde{R}_b(\ell, \omega),$$

and the remainder satisfies

$$\sup\{\|\tilde{R}_b(\ell, \omega)\|_{\mathcal{L}(L_2(\Sigma))} : \omega \in K \wedge \ell \in (0, \ell_0)\} < \infty.$$

Recalling that the operator \mathcal{Q}_0 is invertible, we obtain

$$\ell \mathcal{Q}_b(\ell, \omega) = \mathcal{Q}_0 \left(I + \ell \mathcal{Q}_0^{-1} \tilde{R}_b(\ell, \omega) \right).$$

Choosing $\ell > 0$ sufficiently small implies that $\mathcal{Q}_b(\ell, \omega)$ is invertible for all $\omega \in K$ and $\ell \in (0, \ell_0)$. In particular, 0 cannot be an eigenvalue of $\mathcal{Q}_b(\ell, \omega)$. As a consequence the discrete eigenvalues of the operator $A_{\ell,b}$ converge to π^2/α^2 as $\ell \rightarrow 0$.

In what follows we consider for real ω not only the kernel of the operator $\mathcal{Q}_b(\ell, \omega)$, but more generally the discrete eigenvalues of the self-adjoint realisation of $\mathcal{Q}_b(\ell, \omega)$. For $\ell > 0$ and $\omega \in \mathbb{R} \setminus [\pi^2/\alpha^2, \infty)$ we denote

$$\mu_1(\ell, \omega) \leq \mu_2(\ell, \omega) \leq \dots$$

these eigenvalues counted with multiplicities.

Lemma 2.9. *Let $\ell_0 > 0$ and $\varepsilon > 0$ be chosen as in Theorem 2.6. Then the following assertions hold true:*

- (1) *For fixed $\ell > 0$ the function $\mu_1(\ell, \cdot)$ is strictly decreasing in $(-\infty, \pi^2/\alpha^2)$.*
- (2) *For fixed $\ell \in (0, \ell_0)$ we have $\mu_1(\ell, \omega) \rightarrow -\infty$ as $\omega \rightarrow \pi^2/\alpha^2$.*

- (3) For fixed $\omega \in (\pi^2/\alpha^2 - \varepsilon, \pi^2/\alpha^2)$ we have $\mu_1(\ell, \omega) \rightarrow \infty$ as $\ell \rightarrow 0$.
 (4) There exists $\tilde{\ell}_0 = \tilde{\ell}_0(\alpha, \Sigma)$ such that for all $\tilde{\ell} \in (0, \ell_0)$ and for all $|\omega - \pi^2/\alpha^2| < \varepsilon$ we have $\mu_2(\ell, \omega) > 0$.

Proof. We note that for fixed $\xi \in \mathbb{R}$ the function $m_\omega(\xi)$ is strictly decreasing in ω as can easily be seen from

$$m_\omega(\xi) = \frac{1}{\alpha} + \sum_{k=1}^{\infty} \frac{2\alpha(\xi^2 - \omega)}{\alpha^2(\xi^2 - \omega) + k^2\pi^2} = \frac{1}{\alpha} + \sum_{k=1}^{\infty} \frac{2\alpha}{\alpha^2 + \frac{k^2\pi^2}{\xi^2 - \omega}}.$$

Thus, for $-\infty < \omega_1 < \omega_2 < \pi^2/\alpha^2$ and $g \in \tilde{H}_0^{1/2}(\Sigma) \setminus \{0\}$ we have

$$q_b(\ell, \omega_1)[g] > q_b(\ell, \omega_2)[g].$$

Then assertion (1) follows by applying the min-max principle for self-adjoint operators. Let us now prove assertion (2). Decomposition (2.48) and the min-max principle for self-adjoint operators yield for $\ell \in (0, \ell_0)$ and $|\omega - \pi^2/\alpha^2| < \varepsilon$ that $\mu_1(\ell, \omega) \leq q_b(\ell, \omega)[g_0]$ for any $g_0 \in \tilde{H}_0^{1/2}(\Sigma)$ with $\|g_0\|_{L_2(\Sigma)} = 1$. Choosing g_0 such that $\langle P_{\text{ct}}g_0, g_0 \rangle_\Sigma \neq 0$ we obtain

$$\mu_1(\ell, \omega) \leq \frac{1}{\ell} \langle \mathcal{Q}_0 g_0, g_0 \rangle_\Sigma - \ell \cdot \left(\frac{|\Sigma| \cdot \pi^2}{\alpha^3} \right) \cdot \frac{1}{\sqrt{\pi^2/\alpha^2 - \omega}} \cdot \langle P_{\text{ct}}g_0, g_0 \rangle_\Sigma + C_1,$$

which tends to $-\infty$ as $\omega \rightarrow \pi^2/\alpha^2$. Here $C_1 := \sup\{\|R_b(\ell, \omega)\|_{\mathcal{L}(L_2(\Sigma))} : \ell \in (0, \ell_0) \wedge |\omega - \pi^2/\alpha^2| < \varepsilon\}$. This proves (2). To deduce (3) we recall that \mathcal{Q}_0 is invertible and we have $q_0[g] = \langle \mathcal{Q}_0 g, g \rangle_\Sigma \geq 0$ for all $g \in \tilde{H}_0^{1/2}(\Sigma)$. Thus, there exists $\mu_* > 0$ such that

$$\langle \mathcal{Q}_0 g, g \rangle_\Sigma = q_0[g] \geq \mu_* \|g\|_{L_2(\Sigma)}^2, \quad g \in H_0^{1/2}.$$

We note that the spectrum of the self-adjoint realisation of \mathcal{Q}_0 is discrete since the form domain of q_0 is compactly embedded in $L_2(\Sigma)$. Thus, for fixed $\omega \in \mathbb{R} \setminus [\pi^2/\alpha^2, \infty)$ with $|\omega - \pi^2/\alpha^2| < \varepsilon$ we have

$$\mu_1(\ell, \omega) = \inf\{q_b(\ell, \omega)[g] : g \in \tilde{H}_0^{1/2}(\Sigma) \wedge \|g\|_{L_2(\Sigma)} = 1\} \geq \frac{\mu_*}{\ell} - C_1 \rightarrow \infty$$

as $\ell \rightarrow 0$. This proves (3). Assertion (4) follows if we prove that the form $q_b(\ell, \omega)$ is positive on a subset of codimension 1. Choose $g \in \tilde{H}_0^{1/2}(\Sigma)$, $\|g\|_{L_2(\Sigma)} = 1$, orthogonal to the constant functions. Then

$$q_b(\ell, \omega)[g] = \frac{1}{\ell} q_0[g] + \langle R_b(\ell, \omega)g, g \rangle_\Sigma \geq \frac{\mu_*}{\ell} - C_1 > 0$$

for $0 < \ell < \tilde{\ell}_0 := \min\{1, \mu_*/C_1\}$ and $|\omega - \pi^2/\alpha^2| < \varepsilon$. This concludes the proof of Lemma 2.9. \square

Lemma 2.10. *There exists $\ell_0 = \ell_0(\alpha, b, \Sigma) > 0$ such that for all $\ell \in (0, \ell_0)$ the operator $A_{\ell, b}$ has a unique eigenvalue $\lambda(\ell)$ below its essential spectrum.*

Proof. We start by proving the uniqueness of the eigenvalue. Let $\varepsilon > 0$ be chosen as in Theorem 2.6 and Lemma 2.9. Using the remark before Lemma 2.9 we may choose $\ell_0 > 0$ such that $\inf \sigma(A_{\ell, b}) \geq \pi^2/\alpha^2 - \varepsilon$ for all $\ell \in (0, \ell_0)$. Moreover, we assume that

$\mu_2(\ell, \omega) > 0$ for all $\ell \in (0, \ell_0)$ and $\omega \in (\pi^2/\alpha^2 - \varepsilon, \pi^2/\alpha^2)$. Fix $\ell \in (0, \ell_0)$ and assume that $\omega \in \sigma_d(A_{\ell,b})$. Then $\mu_1(\ell, \omega) = 0$. Lemma 2.9 (1) implies for $\omega_1 < \omega < \omega_2 < \pi^2/\alpha^2$

$$\mu_1(\ell, \omega_1) < \mu_1(\ell, \omega) = 0 < \mu_1(\ell, \omega_2).$$

In particular we have $\ker \mathcal{Q}_b(\ell, \omega_1) = \ker \mathcal{Q}_b(\ell, \omega_2) = \{0\}$, which proves the uniqueness of the eigenvalue of $A_{\ell,b}$.

Next we prove the existence of the eigenvalue. Using Lemma 2.9 (3) we may assume that $\mu_1(\ell, \pi^2/\alpha^2 - \varepsilon/2) > 0$ for all $\ell \in (0, \ell_0)$. Fix $\ell \in (0, \ell_0)$. Since $\mu_1(\ell, \omega) \rightarrow -\infty$ as $\omega \rightarrow \pi^2/\alpha^2$ and $\mu_1(\ell, \omega)$ depends continuously on ω it follows that there exists $\tilde{\omega} \in (\pi^2/\alpha^2 - \varepsilon/2, \pi^2/\alpha^2)$ such that $\mu_1(\ell, \tilde{\omega}) = 0$. \square

Remark. Another method of proof for Lemma 2.10 may be based on a variant of operator-valued Rouché's theorem, cf. e.g. [19] or the monograph [1].

Next we prove the asymptotic formula for the eigenvalue of $A_{\ell,b}$ using the Birman-Schwinger principle. To this end we choose $\ell_0 > 0$ such that the operator $\mathcal{Q}_0 + \ell R_b(\ell, \omega)$ is invertible for all $\ell \in (0, \ell_0)$ and $\omega \in (\pi^2/\alpha^2 - \varepsilon, \pi^2/\alpha^2)$. The existence of such an ℓ_0 follows from the estimate (2.49).

Lemma 2.11 (Birman-Schwinger principle). *Let $\ell \in (0, \ell_0)$ and $\omega \in (\pi^2/\alpha^2 - \varepsilon, \pi^2/\alpha^2)$. Then 0 is an eigenvalue of the operator*

$$\ell \mathcal{Q}_b(\ell, \omega) = \mathcal{Q}_0 - \ell^2 \left(\frac{|\Sigma| \cdot \pi^2}{\alpha^3} \right) \frac{1}{\sqrt{\pi^2/\alpha^2 - \omega}} P_{\text{ct}} + \ell R_b(\ell, \omega)$$

if and only if 1 is an eigenvalue of the Birman-Schwinger operator

$$\ell^2 \left(\frac{|\Sigma| \cdot \pi^2}{\alpha^3} \right) \frac{1}{\sqrt{\pi^2/\alpha^2 - \omega}} \cdot P_{\text{ct}}^{1/2} (\mathcal{Q}_0 + \ell R_b(\ell, \omega))^{-1} P_{\text{ct}}^{1/2} \quad (2.50)$$

A proof may be found e.g. in [4].

Since the projection P_{ct} is a rank-one operator with $P_{\text{ct}}^2 = P_{\text{ct}} = P_{\text{ct}}^{1/2}$, the Birman-Schwinger principle implies that ω is an eigenvalue of the operator $A_{\ell,b}$ if and only if the trace of the Birman-Schwinger operator is equal to one, i.e.,

$$\ell^2 \left(\frac{\pi^2}{\alpha^3} \right) \frac{1}{\sqrt{\pi^2/\alpha^2 - \omega}} \langle (\mathcal{Q}_0 + \ell R_b(\ell, \omega))^{-1} \phi_0, \phi_0 \rangle_{\Sigma} = 1,$$

where $\phi_0(x) = 1$ is the (non-normalised) constant function on $L_2(\Sigma)$. For the choice $\omega = \lambda(\ell)$ we obtain

$$\sqrt{\pi^2/\alpha^2 - \lambda(\ell)} = \ell^2 \left(\frac{\pi^2}{\alpha^3} \right) \langle (\mathcal{Q}_0 + \ell R_b(\ell, \lambda(\ell)))^{-1} \phi_0, \phi_0 \rangle_{\Sigma}. \quad (2.51)$$

Next we use an asymptotic expansion for the resolvent term. We have

$$\begin{aligned} (\mathcal{Q}_0 + \ell R_b(\ell, \omega))^{-1} &= (I + \ell \mathcal{Q}_0^{-1} R_b(\ell, \omega))^{-1} \mathcal{Q}_0^{-1}, \\ &= \sum_{k=0}^{\infty} \ell^k (-\mathcal{Q}_0^{-1} R_b(\ell, \omega))^k \mathcal{Q}_0^{-1} = \mathcal{Q}_0^{-1} + \mathcal{O}(\ell), \end{aligned} \quad (2.52)$$

uniformly in $\omega \in (\pi^2/\alpha^2, -\varepsilon, \pi^2/\alpha^2)$. Note that for sufficiently small ℓ the sum converges absolutely in $\mathcal{L}(L_2(\Sigma))$. Hence,

$$\langle (\mathcal{Q}_0 + \ell R_b(\ell, \lambda(\ell)))^{-1} \phi_0, \phi_0 \rangle_\Sigma = \langle \mathcal{Q}_0^{-1} \phi_0, \phi_0 \rangle_\Sigma + \mathcal{O}(\ell)$$

as $\ell \rightarrow 0$, and thus,

$$\begin{aligned} \sqrt{\pi^2/\alpha^2 - \lambda(\ell)} &= \ell^2 \left(\frac{\pi^2}{\alpha^3} \right) \langle (\mathcal{Q}_0 + \ell R_b(\ell, \lambda(\ell)))^{-1} \phi_0, \phi_0 \rangle_\Sigma \\ &= \left(\frac{\pi^2}{\alpha^3} \right) \cdot \langle \mathcal{Q}_0^{-1} \phi_0, \phi_0 \rangle_\Sigma \cdot \ell^2 + \mathcal{O}(\ell^3). \end{aligned}$$

This proves the first term of the asymptotics in Theorem 2.7 with

$$\tau_0(\Sigma) := \langle \mathcal{Q}_0^{-1} \phi_0, \phi_0 \rangle_\Sigma = \langle \mathcal{Q}_0^{-1/2} \phi_0, \mathcal{Q}_0^{-1/2} \phi_0 \rangle_\Sigma > 0. \quad (2.53)$$

Until now no additional assumptions on $b \in L_\infty(\mathbb{R})$ were necessary. Now let b be differentiable in a neighbourhood of 0. Then

$$\langle b T_\ell g, T_\ell h \rangle_{\Sigma_\ell} = \int_\Sigma b(\ell x) \cdot g(x) \overline{h(x)} \, dx = b(0) \langle g, h \rangle_\Sigma + \mathcal{O}(\ell).$$

Note that the remainder may be estimated uniformly in the operator norm. Thus, together with Theorem 2.6 we obtain

$$R_b(\ell, \omega) = b(0)I + \mathcal{O}(\ell \ln \ell).$$

The estimate holds uniformly in $\ell \in (0, \ell_0)$ and $|\omega - \pi^2/\alpha^2| < \varepsilon$. Using Formula (2.52) we obtain

$$\begin{aligned} (\mathcal{Q}_0 + \ell R_b(\ell, \lambda(\ell)))^{-1} &= \mathcal{Q}_0^{-1} + \ell \mathcal{Q}_0^{-1} R_b(\ell, \lambda(\ell)) \mathcal{Q}_0^{-1} + \mathcal{O}(\ell^2) \\ &= \mathcal{Q}_0^{-1} + \ell \cdot b(0) \cdot I + \mathcal{O}(\ell^2 \cdot \ln \ell). \end{aligned}$$

Finally,

$$\begin{aligned} \sqrt{\pi^2/\alpha^2 - \lambda(\ell)} &= \ell^2 \left(\frac{\pi^2}{\alpha^3} \right) \langle (\mathcal{Q}_0 + \ell R_b(\ell, \omega))^{-1} \phi_0, \phi_0 \rangle_\Sigma \\ &= \left(\frac{\pi^2}{\alpha^3} \right) \cdot \tau_0(\Sigma) \cdot \ell^2 + \left(\frac{b(0)\pi^2}{\alpha^3} \right) \cdot \tau_1(\Sigma) \cdot \ell^3 + \mathcal{O}(\ell^4 \ln \ell) \end{aligned}$$

with

$$\tau_1(\Sigma) := \langle \mathcal{Q}_0^{-2} \phi_0, \phi_0 \rangle_\Sigma = \langle \mathcal{Q}_0^{-1} \phi_0, \mathcal{Q}_0^{-1} \phi_0 \rangle_\Sigma > 0. \quad (2.54)$$

This proves Theorem 2.7.

Now let $\Sigma = (-1, 1)$. Then the operator \mathcal{Q}_0 becomes the composition of the standard finite Hilbert transform and the derivative. Using [2, Formula (4.8)] or [1, Section 5.2] we obtain

$$(\mathcal{Q}_0^{-1} \phi_0)(x) = \sqrt{1 - x^2},$$

which implies

$$\langle \mathcal{Q}_0^{-1} \phi_0, \phi_0 \rangle_\Sigma = \int_{-1}^1 \sqrt{1 - x^2} \, dx = \frac{\pi}{2}.$$

For the sake of simplicity we assume that $b \in C^2(\mathbb{R})$. Then

$$\langle bT_\ell g, T_\ell h \rangle_{\Sigma_\ell} = \int_{\Sigma} b(\ell x) \cdot g(x) \overline{h(x)} \, dx = b(0) \langle g, h \rangle_{\Sigma} + \ell \cdot b'(0) \langle M_x g, h \rangle_{\Sigma} + \mathcal{O}(\ell^2),$$

where $M_x : L_2(\Sigma) \rightarrow L_2(\Sigma)$ is the multiplication operator $(M_x f)(x) = x f(x)$. Theorem 2.6 implies

$$\begin{aligned} R_b(\ell, \omega) &= b(0) \cdot I + \ell \ln \ell \cdot \frac{\pi}{\alpha^2} P_{\text{ct}} + \ell \left(\frac{\rho_0(\alpha)}{\pi} P_{\text{ct}} + \frac{\pi}{2\alpha^2} K_{\ln|x|} + b'(0) M_x \right) \\ &\quad + R_b^{(1)}(\ell, \omega), \end{aligned} \quad (2.55)$$

with

$$\|R_b^{(1)}(\ell, \lambda(\ell))\|_{\mathcal{L}(L_2(\Sigma))} \leq C(\ell^3 \ln \ell + \pi^2/\alpha^2 - \lambda(\ell)) = \mathcal{O}(\ell^3 \ln \ell).$$

To calculate the asymptotic behaviour of the eigenvalue we use the expansion

$$\begin{aligned} (\mathcal{Q}_0 + \ell R_b(\ell, \lambda(\ell)))^{-1} &= \mathcal{Q}_0^{-1} - \ell \cdot b(0) \mathcal{Q}_0^{-2} - \ell^2 \ln \ell \cdot \frac{\pi}{\alpha^2} \mathcal{Q}_0^{-1} P_{\text{ct}} \mathcal{Q}_0^{-1} \\ &\quad - \ell^2 \cdot \mathcal{Q}_0^{-1} \left(\frac{\rho_0(\alpha)}{\pi} P_{\text{ct}} + \frac{\pi}{2\alpha^2} K_{\ln|x|} + b'(0) M_x \right) \mathcal{Q}_0^{-1} \\ &\quad + \ell^2 \cdot b(0)^2 \mathcal{Q}_0^{-3} + \mathcal{O}(\ell^3 \ln \ell). \end{aligned}$$

Then

$$\begin{aligned} &\ell^{-2} \left(\frac{\pi^2}{\alpha^3} \right)^{-1} \cdot \sqrt{\pi^2/\alpha^2 - \lambda(\ell)} \\ &= \frac{\pi}{2} - \ell \cdot b(0) \langle \mathcal{Q}_0^{-2} \phi_0, \phi_0 \rangle_{(-1,1)} - \ell^2 \ln \ell \cdot \frac{\pi}{\alpha^2} \langle \mathcal{Q}_0^{-1} P_{\text{ct}} \mathcal{Q}_0^{-1} \phi_0, \phi_0 \rangle_{(-1,1)} \\ &\quad - \ell^2 \cdot \frac{\rho_0(\alpha)}{\pi} \langle \mathcal{Q}_0^{-1} P_{\text{ct}} \mathcal{Q}_0^{-1} \phi_0, \phi_0 \rangle_{(-1,1)} - \ell^2 \cdot \frac{\pi}{\alpha^2} \langle \mathcal{Q}_0^{-1} K_{\ln|x|} \mathcal{Q}_0^{-1} \phi_0, \phi_0 \rangle_{(-1,1)} \\ &\quad - \ell^2 \cdot b'(0) \langle \mathcal{Q}_0^{-1} M_x \mathcal{Q}_0^{-1} \phi_0, \phi_0 \rangle_{(-1,1)} + \ell^2 \cdot b(0)^2 \langle \mathcal{Q}_0^{-3} \phi_0, \phi_0 \rangle_{(-1,1)} \\ &\quad + \mathcal{O}(\ell^3 \ln \ell). \end{aligned}$$

In order to calculate the asymptotic behaviour of $\langle (\mathcal{Q}_0^{-1} + \ell R_{\ell, \lambda(\ell)})^{-1} \phi_0, \phi_0 \rangle_{(-1,1)}$ we shall need the following identities

$$\begin{aligned} \langle \mathcal{Q}_0^{-2} \phi_0, \phi_0 \rangle_{(-1,1)} &= \langle \mathcal{Q}_0^{-1} \phi_0, \mathcal{Q}_0^{-1} \phi_0 \rangle_{(-1,1)} = \int_{-1}^1 (1 - x^2) \, dx = \frac{4}{3}, \\ \langle \mathcal{Q}_0^{-1} P_{\text{ct}} \mathcal{Q}_0^{-1} \phi_0, \phi_0 \rangle_{(-1,1)} &= \langle P_{\text{ct}} \mathcal{Q}_0^{-1} \phi_0, \mathcal{Q}_0^{-1} \phi_0 \rangle_{(-1,1)} = \frac{\pi^2}{8}, \\ \langle \mathcal{Q}_0^{-1} M_x \mathcal{Q}_0^{-1} \phi_0, \phi_0 \rangle_{(-1,1)} &= \langle M_x \mathcal{Q}_0^{-1} \phi_0, \mathcal{Q}_0^{-1} \phi_0 \rangle_{(-1,1)} = \int_{-1}^1 x(1 - x^2) \, dx = 0. \end{aligned}$$

Next we calculate $\langle K_{\ln|x|} \mathcal{Q}_0^{-1} \phi_0, \mathcal{Q}_0^{-1} \phi_0 \rangle_{(-1,1)}$. Recall that $(\mathcal{Q}_0^{-1} \phi_0)(x) = \sqrt{1-x^2}$. Using [1, Formulae (5.6)-(5.9)] for $\psi := K_{\ln|x|} \mathcal{Q}_0^{-1} \phi_0$ we obtain

$$\psi'(x) = \text{p.v.} \int_{-1}^1 \frac{\phi_0(x)}{x-y} dy = \text{p.v.} \int_{-1}^1 \frac{\sqrt{1-x^2}}{x-y} dy = \pi x.$$

Thus, $\psi(x) = \frac{\pi x^2}{2} + \psi(0)$, where

$$\psi(0) = \int_{-1}^1 \ln|x| \sqrt{1-x^2} dx = 2 \int_0^1 \ln|x| \sqrt{1-x^2} dx = -\frac{\pi}{2} \left(\frac{1}{2} + \ln 2 \right),$$

cf. [20, Section 4.241]. Hence,

$$\begin{aligned} & \langle K_{\ln|x|} \mathcal{Q}_0^{-1} \phi_0, \mathcal{Q}_0^{-1} \phi_0 \rangle_{(-1,1)} \\ &= \frac{\pi}{2} \int_{-1}^1 x^2 \sqrt{1-x^2} dx - \frac{\pi}{2} \left(\frac{1}{2} + \ln 2 \right) \int_{-1}^1 \sqrt{1-x^2} dx = \frac{\pi^2}{16} (-1 - \ln 16). \end{aligned}$$

Setting

$$\begin{aligned} \rho_1 &:= \langle \mathcal{Q}_0^{-3} \phi_0, \phi_0 \rangle_{(-1,1)} = \langle \mathcal{Q}_0^{-2} \phi_0, \mathcal{Q}_0^{-1} \phi_0 \rangle_{(-1,1)} \\ &= \langle \mathcal{Q}_0^{-3/2} \phi_0, \mathcal{Q}_0^{-3/2} \phi_0 \rangle_{(-1,1)} > 0 \end{aligned} \tag{2.56}$$

we obtain

$$\begin{aligned} \sqrt{\pi^2/\alpha^2 - \lambda(\ell)} &= \ell^2 \left(\frac{\pi^3}{2\alpha^3} \right) - \ell^3 \left(\frac{4b(0)\pi^2}{3\alpha^3} \right) - \ell^4 \ln \ell \left(\frac{\pi^5}{8\alpha^5} \right) \\ &\quad - \ell^4 \left(\frac{\rho_0(\alpha)\pi^3}{8\alpha^3} - \frac{\pi^5}{32\alpha^5} (1 + \ln 16) - b(0)^2 \rho_1 \cdot \frac{\pi^2}{\alpha^3} \right) + \mathcal{O}(\ell^5 \ln \ell). \end{aligned}$$

This proves Theorem 2.8.

Concluding the two-dimensional case we briefly want to sketch what happens in the case of two waveguides of width α_+ and α_- , which are coupled through a window $\Sigma_\ell := \ell \cdot \Sigma$. We use the same ansatz and introduce the corresponding Dirichlet-to-Neumann operators $D_{\ell,\omega}^+$ and $D_{\ell,\omega}^-$ on the upper and on the lower waveguide. Comparing the normal derivatives along the window we observe that ω is an eigenvalue of the corresponding Dirichlet-Laplacian if and only if 0 is an eigenvalue of

$$D_{\ell,\omega} := D_{\ell,\omega}^+ + D_{\ell,\omega}^-.$$

Using the same scaling operator T_ℓ as above leads to the analysis of the operator

$$\mathcal{Q}(\ell, \omega) = \mathcal{Q}^+(\ell, \omega) + \mathcal{Q}^-(\ell, \omega) = T_\ell^* D_{\ell,\omega}^+ T_\ell + T_\ell^* D_{\ell,\omega}^- T_\ell$$

In what follows we assume that $\alpha_+ > \alpha_-$ so that the essential spectrum of the corresponding operator A_ℓ is given by the interval $[\pi^2/\alpha_+^2, \infty)$. Using the asymptotic expansions of $\mathcal{Q}^\pm(\ell, \omega)$ as $\ell \rightarrow 0$ and $\omega \rightarrow \pi^2/\alpha_+^2$ we obtain

$$\mathcal{Q}(\ell, \omega) = \frac{2}{\ell} \mathcal{Q}_0 - \frac{\ell}{\sqrt{\pi^2/\alpha^2 - \omega}} \left(\frac{|\Sigma| \cdot \pi^2}{\alpha_+^3} \right) P_{\text{ct}} + \mathcal{O}(1),$$

The same approach yields now the following result.

Theorem 2.12. *In the case of two coupled waveguides the ground state eigenvalue $\lambda(\ell)$ satisfies*

$$\sqrt{\pi^2/\alpha^2 - \lambda(\ell)} = \left(\frac{\pi^2}{2\alpha_+^3} \right) \cdot \tau_0(\Sigma) \cdot \ell^2 + \mathcal{O}(\ell^3) \quad \text{as } \ell \rightarrow 0. \quad (2.57)$$

Here $\tau_0(\Sigma) > 0$ is again given by (2.53).

3. INFINITE LAYERS

We consider the mixed problem for an infinite layer $\Omega := \mathbb{R}^2 \times (0, \alpha)$ with coordinates $(x, y) = (x_1, x_2, y) \in \mathbb{R}^2 \times (0, \alpha)$. Let $\Sigma \times \{0\} \subseteq \partial\Omega$ be the Robin window, where $\Sigma \subseteq \mathbb{R}^2$ is a bounded open subset with Lipschitz boundary. For $\ell > 0$ we denote by $\Sigma_\ell := \ell \cdot \Sigma \subseteq \mathbb{R}^2$ the scaled window. Let $b \in L_\infty(\mathbb{R}^2)$ be a real-valued function and consider the quadratic form

$$a_{\ell,b}[u] := \int_{\Omega} |u(x, y)|^2 \, dx \, dy + \int_{\Sigma_\ell} b(x) \cdot |u(x, 0)|^2 \, dx \quad (3.1)$$

with the form domain

$$D[a_{\ell,b}] := \{u \in H^1(\Omega) : u|_{\mathbb{R}^2 \times \{\alpha\}} = 0 \wedge \text{supp}(u|_{\mathbb{R}^2 \times \{0\}}) \subseteq \overline{\Sigma_\ell}\}. \quad (3.2)$$

As in the two-dimensional case we observe that $a_{\ell,b}$ is a closed semi-bounded form in $L_2(\Omega)$, and thus, it induces a self-adjoint operator $A_{\ell,b}$. The essential spectrum of $A_{\ell,b}$ is independent of b and ℓ and given by $\sigma_{\text{ess}}(A_{\ell,b}) = [\pi^2/\alpha^2, \infty)$. We prove the following theorem.

Theorem 3.1. *There exists $\ell_0 = \ell_0(\alpha, b, \Sigma) > 0$ such that the operator $A_{\ell,b}$ has a unique eigenvalue $\lambda(\ell)$ below the essential spectrum $[\pi^2/\alpha^2, \infty)$. If b is C^1 in some neighbourhood of $0 \in \mathbb{R}^2$ then the eigenvalue satisfies the asymptotic estimate*

$$\ln(\pi^2/\alpha^2 - \lambda(\ell)) = -\ell^{-3} \frac{4\alpha^3}{\tau_0(\Sigma) + \tau_1(\Sigma)b(0)\ell + \mathcal{O}(\ell^2)} \quad \text{as } \ell \rightarrow 0, \quad (3.3)$$

with constants $\tau_0(\Sigma) > 0$ given by (3.17) and $\tau_1(\Sigma) > 0$ given by (3.18).

Since we shall only slightly modify our approach we will merely sketch the major steps of the proof. Actually, most of the results proven in the two-dimensional case may be reused. Let $\omega \in \mathbb{C}$ and $g \in H^{1/2}(\mathbb{R}^2)$. We consider for $u \in H^1(\Omega)$ the Poisson problem

$$(-\Delta - \omega)u = 0 \quad \text{in } \Omega, \quad u(\cdot, 0) = g, \quad u(\cdot, \alpha) = 0. \quad (3.4)$$

Applying the Fourier transform with respect to the first two variables leads for every $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ to the following Sturm-Liouville problem

$$(-\partial_y^2 + |\xi|^2 - \omega)\hat{u}(\xi, \cdot) = 0, \quad \text{in } (0, \alpha) \quad \hat{u}(\xi, 0) = \hat{g}(\xi), \quad \hat{u}(\xi, \alpha) = 0,$$

where $\xi \in \mathbb{R}^2$. The solution of (3.4) is given by

$$\hat{u}(\xi, y) := \hat{g}(\xi) \cdot \frac{\sinh((\alpha - y)\sqrt{|\xi|^2 - \omega})}{\sinh(\alpha\sqrt{|\xi|^2 - \omega})}, \quad (3.5)$$

which is similar to Formula (2.5). In the same way we obtain that the Poisson problem (3.4) is uniquely solvable for all $g \in H^{1/2}(\mathbb{R}^2)$ if and only if $\omega \in \mathbb{C} \setminus [\pi^2/\alpha^2, \infty)$. Moreover, there exists a constant $c = c(\alpha, \omega) > 0$ such that $\|u\|_{H^1(\Omega)} \leq c\|g\|_{H^{1/2}(\mathbb{R}^2)}$. In what follows let $\omega \in \mathbb{C} \setminus [\pi^2/\alpha^2, \infty)$. Then the normal derivative of u satisfies

$$\widehat{\partial_n u}(\xi, 0) = m_\omega(|\xi|) \cdot \hat{g}(\xi), \quad \xi \in \mathbb{R}^2,$$

where the function m_ω is defined as in the two-dimensional case, i.e.,

$$m_\omega(|\xi|) = \sqrt{|\xi|^2 - \omega} \cdot \coth(\alpha \sqrt{|\xi|^2 - \omega}). \quad (3.6)$$

Hence, the Dirichlet-to-Neumann operator for the infinite layer is given by the Fourier integral operator

$$D_\omega : H^{1/2}(\mathbb{R}^2) \rightarrow H^{-1/2}(\mathbb{R}^2), \quad \widehat{D_\omega g}(\xi) := m_\omega(|\xi|) \cdot \hat{g}(\xi). \quad (3.7)$$

The next step is to define the truncated operator on the boundary. The corresponding spaces $\tilde{H}_0^{1/2}(\Sigma_\ell)$ and $H^{-1/2}(\Sigma_\ell)$ are defined as in (2.9) and (2.10). As both Σ and Σ_ℓ have Lipschitz boundary, the dual pairing (2.11) still holds true, cf. [25, Theorems 3.14, 3.30]. Put

$$D_{\ell, \omega} + b : \tilde{H}_0^{1/2}(\Sigma_\ell) \rightarrow H^{-1/2}(\Sigma_\ell), \quad D_{\ell, \omega} + b := r_\ell(D_\omega + b)e_\ell, \quad (3.8)$$

where $r_\ell : H^{-1/2}(\mathbb{R}^2) \rightarrow H^{-1/2}(\Sigma_\ell)$ denotes the restriction operator and $e_\ell : \tilde{H}_0^{1/2}(\Sigma_\ell) \rightarrow H^{1/2}(\mathbb{R}^2)$ the embedding operator. As in Lemma 2.3 we obtain

$$\dim \ker(A_{\ell, b} - \omega) = \dim \ker(D_{\ell, \omega} + b). \quad (3.9)$$

Let

$$T_\ell : L_2(\Sigma) \rightarrow L_2(\Sigma_\ell), \quad (T_\ell g)(x) := \ell^{-1}g(x/\ell).$$

be the unitary scaling operator. In what follows we consider the scaled operator

$$\mathcal{Q}_b(\ell, \omega) = T_\ell^*(D_{\ell, \omega} + b)T_\ell \quad (3.10)$$

together with its associated sesquilinear form

$$q_b(\ell, \omega) := \langle \mathcal{Q}_b(\ell, \omega)g, h \rangle_\Sigma \quad (3.11)$$

$$= \ell^2 \int_{\mathbb{R}^2} m_\omega(|\xi|) \cdot \hat{g}(\ell\xi) \overline{\hat{h}(\ell\xi)} \, d\xi + \int_\Sigma b(\ell x) \cdot g(x) \overline{h(x)} \, dx, \quad (3.12)$$

where $g, h \in D[q_b(\ell, \omega)] := \tilde{H}_0^{1/2}(\Sigma)$. As before we define $\mathcal{Q}_0 : \tilde{H}_0^{1/2}(\Sigma) \rightarrow H^{-1/2}(\Sigma)$,

$$\langle \mathcal{Q}_0 g, h \rangle_\Sigma := q_0[g, h] := \int_{\mathbb{R}^2} |\xi| \cdot \hat{g}(\xi) \cdot \overline{\hat{h}(\xi)} \, d\xi, \quad (3.13)$$

and let P_{ct} denote the projection onto the space of constant functions in $L_2(\Sigma)$. Moreover, we denote by $K_{\frac{1}{|\cdot|}} : L_2(\Sigma) \rightarrow L_2(\Sigma)$ the following convolution operator

$$(K_{\frac{1}{|\cdot|}} f)(z) := \int_\Sigma \frac{f(z)}{|x - z|} \, dx, \quad z \in \Sigma. \quad (3.14)$$

Theorem 3.2. *Let $b = 0$. There exists $\ell_0 > 0$ and $\varepsilon > 0$ such that for $\ell \in (0, \ell_0)$ and $|\omega - \pi^2/\alpha^2| < \varepsilon$ the following expansion holds true*

$$\mathcal{Q}_0(\ell, \omega) = \frac{1}{\ell} \mathcal{Q}_0 + \ell^2 \cdot \frac{|\Sigma|}{4\alpha^3} \ln(\pi^2/\alpha^2 - \omega) P_{\text{ct}} - \ell \frac{\pi}{4\alpha^2} K_{\frac{1}{|\Sigma|}} + R(\ell, \omega). \quad (3.15)$$

Here $|\Sigma|$ denotes the volume of Σ and the remainder satisfies

$$\|R(\ell, \omega)\|_{\mathcal{L}(L_2(\Sigma))} \leq C(\ell^2 + \pi^2/\alpha^2 - \omega)$$

for some constant $C = C(\alpha, \Sigma) > 0$ which is independent of ℓ, ω .

Proof. We use the same decomposition for $q_0(\ell, \omega)$ as in the two-dimensional case and put

$$\begin{aligned} q_0(\ell, \omega)[g, h] &= \ell^2 \left(\int_{\{|\xi| \leq 1\}} + \int_{\{|\xi| > 1\}} \right) m_\omega(|\xi|) \cdot \hat{g}(\ell\xi) \overline{\hat{h}(\ell\xi)} \, d\xi \\ &=: q_0^{(1)}(\ell, \omega)[g, h] + q_0^{(2)}(\ell, \omega)[g, h]. \end{aligned}$$

Recall that

$$m_\omega(\xi) = \frac{1}{\alpha} + \sum_{k=1}^{\infty} \frac{2\alpha(\xi^2 - \omega)}{\alpha^2(\xi^2 - \omega) + k^2\pi^2} = -\left(\frac{2\pi^2}{\alpha^3}\right) \frac{1}{\xi^2 - \omega + \pi^2/\alpha^2} + \mathcal{O}(1)$$

and thus,

$$q_0^{(1)}(\ell, \omega)[g, h] = -\ell^2 \cdot \left(\frac{2\pi^2}{\alpha^3}\right) \int_{\{|\xi| \leq 1\}} \frac{1}{|\xi|^2 - \omega + \pi^2/\alpha^2} \cdot \hat{g}(\ell\xi) \overline{\hat{h}(\ell\xi)} \, d\xi + \mathcal{O}(\ell^2).$$

We note that the first expression coincides almost with the free resolvent of the Laplacian in \mathbb{R}^2 , which with respect to the spectral parameter ω has a logarithmic singularity. Using the Taylor expansion of $\hat{g} \cdot \bar{\hat{h}}$ at 0 we have

$$\begin{aligned} & -\ell^2 \cdot \left(\frac{2\pi^2}{\alpha^3}\right) \int_{\{|\xi| \leq 1\}} \frac{1}{|\xi|^2 - \omega + \pi^2/\alpha^2} \cdot \hat{g}(\ell\xi) \overline{\hat{h}(\ell\xi)} \, d\xi \\ &= -\ell^2 \cdot \left(\frac{2\pi^2}{\alpha^3}\right) \cdot \sum_{\beta \in \mathbb{N}_0^2} \ell^{|\beta|} \frac{1}{\beta!} \frac{\partial^\beta}{\partial \xi^\beta} \left(\hat{g}(\xi) \bar{\hat{h}}(\xi) \right) \Big|_{\xi=0} \cdot \int_{\{|\xi| \leq 1\}} \frac{\xi^{\beta+1}}{|\xi|^2 + \pi^2/\alpha^2 - \omega} \, d\xi \\ &= -\ell^2 \cdot \left(\frac{2\pi^2}{\alpha^3}\right) \hat{g}(0) \bar{\hat{h}}(0) \cdot \int_{\{|\xi| \leq 1\}} \frac{1}{|\xi|^2 + \pi^2/\alpha^2 - \omega} \, d\xi + \mathcal{O}(\ell^3), \end{aligned}$$

since for $|\beta| \geq 1$ we have

$$\left| \int_{\{|\xi| \leq 1\}} \frac{\xi^\beta}{|\xi|^2 + \pi^2/\alpha^2 - \omega} \, d\xi \right| \leq \int_0^1 \frac{r^2}{r^2 + \pi^2/\alpha^2 - \omega} \, dr \leq C$$

and C may be chosen independently of ω . Moreover,

$$\int_{\{|\xi| \leq 1\}} \frac{1}{|\xi|^2 + \pi^2/\alpha^2 - \omega} \, d\xi = -\frac{1}{2} \ln(\pi^2/\alpha^2 - \omega) + \mathcal{O}(1),$$

and thus,

$$\begin{aligned} q_0^{(1)}(\ell, \omega)[g, h] &= \ell^2 \cdot \left(\frac{\pi^2}{\alpha^3} \right) \hat{g}(0) \cdot \overline{\hat{h}(0)} \cdot \ln(\pi^2/\alpha^2 - \omega) + \mathcal{O}(\ell^3) \\ &= \ell^2 \cdot \left(\frac{|\Sigma|}{4\alpha^3} \right) \ln(\pi^2/\alpha^2 - \omega) \langle P_{\text{ct}} g, h \rangle_\Sigma + \mathcal{O}(\ell^3). \end{aligned}$$

Next we consider the form $q_0^{(2)}(\ell, \omega)$. The expansion (2.40) of m_ω for large $|\xi|$ implies

$$\begin{aligned} q_0^{(2)}(\ell, \omega)[g, h] &= \ell^2 \int_{\{|\xi| > 1\}} m_\omega(|\xi|) \cdot \hat{g}(\ell\xi) \overline{\hat{h}(\ell\xi)} \, d\xi \\ &= \frac{1}{\ell} q_0[g, h] - \ell^2 \cdot \frac{\omega}{2} \int_{\mathbb{R}^2} \frac{\hat{g}(\ell\xi) \overline{\hat{h}(\ell\xi)}}{|\xi|} \, d\xi + \ell^2 \int_{\mathbb{R}^2} m_{\omega, \text{res}}(|\xi|) \cdot \hat{g}(\ell\xi) \overline{\hat{h}(\ell\xi)} \, d\xi, \end{aligned}$$

with

$$m_{\omega, \text{res}}(|\xi|) = -\mathbf{1}_{\{|\xi| \leq 1\}}(\xi) \cdot |\xi| + \mathbf{1}_{\{|\xi| > 1\}}(\xi) \cdot (m_\omega(|\xi|) - |\xi|) + \frac{\omega}{2|\xi|} = \mathcal{O}(|\xi|^{-3}).$$

We choose the functions $X_\omega, X_{\omega, \text{res}} \in C^\infty(\mathbb{R}^2 \setminus \{0\})$ such that

$$\hat{X}_\omega(\xi) = \frac{\omega}{4\pi|\xi|} \quad \text{and} \quad \hat{X}_{\omega, \text{res}}(\xi) = \frac{1}{2\pi} m_{\omega, \text{res}}(\xi).$$

Calculating X_ω for $(s, \varphi) \in \mathbb{R}_+ \times (0, 2\pi)$, $x = (s \cos \varphi, s \sin \varphi)$ we have

$$\begin{aligned} X_\omega(x) &= \frac{\omega}{8\pi^2} \int_{\mathbb{R}^2} \frac{e^{ix\xi}}{|\xi|} \, d\xi = \frac{\omega}{8\pi^2} \int_0^\infty \int_{-\pi}^\pi e^{ist(\cos \varphi \sin \varphi) \cdot (\cos u \sin u)^T} \, du \, dt \\ &= \frac{\omega}{4\pi} \int_0^\infty J_0(ts) \, dt. \end{aligned}$$

Here J_0 is the Bessel function of the first kind of order 0. Moreover, all integrals should be interpreted as oscillatory integrals or improper Riemann integrals. Using [20, Section 6.511] we obtain

$$X_\omega(x) = \frac{\omega}{4\pi|x|} \int_0^\infty J_0(r) \, dr = \frac{\omega}{4\pi|x|}, \quad (3.16)$$

and thus,

$$\begin{aligned} \ell^2 \cdot \frac{\omega}{2} \int_{\mathbb{R}^2} \frac{\hat{g}(\ell\xi) \overline{\hat{h}(\ell\xi)}}{|\xi|} \, d\xi &= \frac{\omega}{4\pi} \langle K_{\frac{1}{|x|}} T_\ell g, T_\ell h \rangle_{\Sigma_\ell} = \frac{\omega \cdot \ell^2}{4\pi} \int_{\Sigma \times \Sigma} \frac{g(x) \overline{h(z)}}{\ell|z - x|} \, dx \, dz \\ &= \ell \cdot \frac{\pi}{4\alpha^2} \langle K_{\frac{1}{|x|}} g, h \rangle_\Sigma + \mathcal{O}(\pi^2/\alpha^2 - \omega). \end{aligned}$$

Note that $m_{\omega, \text{res}}(\xi) = \mathcal{O}(|\xi|^{-3})$ as $|\xi| \rightarrow \infty$, uniformly in $\omega \in (0, \pi^2/\alpha^2)$, and thus,

$$\sup_{\omega \in (0, \pi^2/\alpha^2)} \|X_{\omega, \text{res}}\|_{L^\infty(\mathbb{R}^2)} < \infty,$$

which implies

$$\ell^2 \int_{\mathbb{R}^2} m_{\omega, \text{res}}(|\xi|) \cdot \hat{g}(\ell\xi) \overline{\hat{h}(\ell\xi)} \, d\xi = \ell^2 \int_{\Sigma \times \Sigma} X_{\omega, \text{res}}(\ell(z-x)) \cdot g(x) \overline{h(z)} \, dx \, dz = \mathcal{O}(\ell^2).$$

This concludes the proof of the theorem. \square

Let us now prove the asymptotics of the ground state eigenvalue of the operator $A_{\ell, b}$ as $\ell \rightarrow 0$. We shall omit the proof of the uniqueness or the existence of the eigenvalue for small $\ell > 0$ as this follows in much the same way as in Lemma 2.10. We note that the operator \mathcal{Q}_0 is again invertible and a Fredholm operator since $\tilde{H}_0^{1/2}(\Sigma)$ is compactly embedded into $L_2(\Sigma)$, cf. the arguments from the previous section. Then for arbitrary $b \in L_\infty(\mathbb{R})$ we have

$$\ell \mathcal{Q}_b(\ell, \omega) = \mathcal{Q}_0 + \ell^3 \cdot \frac{|\Sigma|}{4\alpha^3} \ln(\pi^2/\alpha^2 - \omega) \cdot P_{\text{ct}} + \ell R_b(\ell, \omega)$$

with

$$\sup\{\|R_b(\ell, \omega)\| : \ell \in (0, \ell_0) \wedge |\omega - \pi^2/\alpha^2| < \varepsilon\} < \infty.$$

Applying the Birman-Schwinger principle, we obtain the following identity for the eigenvalue $\lambda(\ell)$

$$-\frac{\ell^3}{4\alpha^3} \ln(\pi^2/\alpha^2 - \lambda(\ell)) \cdot \langle (\mathcal{Q}_0 + \ell R_b(\ell, \lambda(\ell)))^{-1} \phi_0, \phi_0 \rangle_\Sigma = 1$$

or equivalently

$$\ln(\pi^2/\alpha^2 - \lambda(\ell)) = -\frac{4\alpha^3}{\ell^3 \cdot \langle (\mathcal{Q}_0 + \ell R_b(\ell, \lambda(\ell)))^{-1} \phi_0, \phi_0 \rangle_\Sigma}.$$

Here $\phi_0(x) = 1$ is again the non-normalised constant function in $L_2(\Sigma)$. As before we obtain

$$-\ln(\pi^2/\alpha^2 - \lambda(\ell)) = \frac{4\alpha^3}{\ell^3 \cdot \tau_1(\Sigma) + \mathcal{O}(\ell^4)}$$

as $\ell \rightarrow 0$. Here

$$\tau_1(\Sigma) := \langle \mathcal{Q}_0^{-1} \phi_0, \phi_0 \rangle_\Sigma = \langle \mathcal{Q}_0^{-1/2} \phi_0, \mathcal{Q}_0^{-1/2} \phi_0 \rangle_\Sigma > 0. \quad (3.17)$$

This proves the first term of the asymptotic formula. Higher terms of the expansion may be calculated as above; assuming smoothness of b we obtain

$$\ln(\pi^2/\alpha^2 - \lambda(\ell)) = -\ell^{-3} \frac{4\alpha^3}{\tau_0(\Sigma) - \ell \cdot \tau_1(\Sigma) \cdot b(0) + \mathcal{O}(\ell^3)} \quad \text{as } \ell \rightarrow 0,$$

where

$$\tau_1(\Sigma) = \langle \mathcal{Q}_0^{-1} \phi_0, \mathcal{Q}_0^{-1} \phi_0 \rangle_\Sigma > 0. \quad (3.18)$$

This concludes the proof of Theorem 3.1.

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